Chapter two

Derivative of functions   مشتقة الدوال

Definition:

Let f be a function, then the derivative of f denoted by f'

Which defined by

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

Remarks:

1) $f'(x)$, $\frac{dy}{d^x}$, $\frac{df(x)}{dx}$, y' are symbols of Derivative.

2) The slope = Derivative   (الميل = المشتقة عند النقطة x)

Example:

Find f' of $f(x) = x^2$ and find the equation of tangent line of $f(x)$ on the point (2,7)

Sol/

by def. $f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$

$$= \lim_{\Delta x \to 0} \frac{(x + \Delta x)^2 - x^2}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{x^2 + 2(x)(\Delta x) + \Delta^2 x - x^2}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{\Delta x(2x + \Delta x)}{\Delta x} = 2x$$

Now, by remark (2) $m = f'(x) = 2x \Rightarrow m = f'(2) = 4$

$$(y - y_1) = m(x - x_1) \rightarrow \text{معادلة المماس}$$

$$(y - 7) = 4(x - 2) \Rightarrow y = 4x - 1$$

Example:

Let $f(x) = \sqrt{x+2}$, by definition find $f'(x)$ and equation of tangent of line at (2,2).

Sol.

By def. $f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$

$$= \lim_{\Delta x \to 0} \frac{\sqrt{x+\Delta x+2} - \sqrt{x+2}}{\Delta x}$$

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$$\lim_{\Delta x \to 0} \frac{\sqrt{x+\Delta x+2} - \sqrt{x+2}}{\Delta x} \cdot \frac{\sqrt{x+\Delta x+2} + \sqrt{x+2}}{\sqrt{x+\Delta x+2} + \sqrt{x+2}}$$

$$\lim_{\Delta x \to 0} \frac{(x+\Delta x+2) - (x+2)}{\Delta x(\sqrt{x+\Delta x+2} + \sqrt{x+2})}$$

$$\lim_{\Delta x \to 0} \frac{x+\Delta x+2 - x - 2}{\Delta x(\sqrt{x+\Delta x+2} + \sqrt{x+2})}$$

$$\lim_{\Delta x \to 0} \frac{\Delta x}{\Delta x(\sqrt{x+\Delta x+2} + \sqrt{x+2})}$$

$$\lim_{\Delta x \to 0} \frac{1}{(\sqrt{x+\Delta x+2} + \sqrt{x+2})}$$

$$= \frac{1}{\sqrt{x+2} + \sqrt{x+2}}$$

$$= \frac{1}{2(\sqrt{x+2})}$$

Since $m = f'(x)$

$$\Rightarrow m = f'(2)$$

$$\Rightarrow m = \frac{1}{2(\sqrt{2+2})}$$

$$= \frac{1}{2\sqrt{4}} = \frac{1}{4}$$

Then

$$y - y_1 = m(x - x_1)$$

$$\Rightarrow (y-2) = \frac{1}{4}(x-2)$$

Home work

1- Let $f(x) = \frac{1}{x}$ find $f'(x)$ and the equation of tangent of line at (3,6).

2- Let $f(x) = x^2 + 5$ find $f'(x)$ and the equation of tangent of line at (2,4).

3- Let $f(x) = x^2 + 2x$ find $f'(x)$ and the equation of tangent of line at (3,2).

4- Let $f(x) = (x^2+1)^2$ find $f'(x)$ and the equation of tangent of line at (-1,4).

5- Let $f(x) = \frac{3}{x^2+4}$ find $f'(x)$ and the equation of tangent of line at (-1,$\frac{3}{5}$).

6- Let $f(x) = \sqrt{x+1}$ find $f'(x)$ and the equation of tangent of line at (3,2).

Definition:

Let f be a function then f is called Differentiable function at the interval [a,b], if $f'(x)$ is exist in this interval. (دالة قابلة للاشتقاق)

Theorem:

If f is differentiable at $x = a$ then f must also be continuous at $x = a$

Example:

$f(x) = x^2$ is diff. and cont. at $x = 2$

Remark:

The converse of the above theorem may not be true in general.

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Example:

$f(x) = |x|$ is cont. at $x = 0$, but it is not diff. at $x = 0$.

Sol/

1- to show that f is cont. at $x = 0$

$$f(0) = |0| = 0$$

$$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+}(x) = 0$$

$$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-}(-x) = 0$$

$$\Rightarrow f(0) = \lim_{x\to 0} f(x) \Rightarrow f \text{ is cont. at } x = 0$$

2- by def.

$$f'(x) = \lim_{\Delta x\to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

$$f'(x) = \lim_{\Delta x \to 0} \frac{|(x + \Delta x)| - |x|}{\Delta x}$$

$$\lim_{\Delta x \to 0} \frac{|x| + |\Delta x| - |x|}{\Delta x}$$

$$\lim_{\Delta x \to 0} \frac{|\Delta x|}{\Delta x} = \begin{bmatrix} \lim\limits_{\Delta x \to 0} \frac{\Delta x}{\Delta x} & = & 1 = L_1 \\ \lim\limits_{\Delta x \to 0} \frac{-\Delta x}{\Delta x} & = & -1 = L_2 \end{bmatrix}, \quad L_1 \neq L_2 \Rightarrow f^{'} \text{ does not exist.}$$

Differentiation Rules قواعد الاشتقاق

Let $f(x)$ and $g(x)$ are two differentiable functions and k is constant member then: -

1. $\frac{d}{dx}\left(kf(x)\right) = k \frac{d}{dx} f(x)$

2. $\frac{d}{dx}\left(f(x) \mp g(x)\right) = \frac{d}{dx} f(x) \mp \frac{d}{dx} g(x)$

3. $\frac{d}{dx}\left(f(x).g(x)\right) = f(x).\frac{d}{dx} g(x) + g(x).\frac{d}{dx} f(x)$

4. $\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{g(x).\frac{d}{dx}f(x) - f(x).\frac{d}{dx}g(x)}{[g(x)]^2}$

5. $if\ f(x) = k \Rightarrow \frac{d}{dx} f(x) = \frac{d}{dx} k = 0$

6. $\frac{d}{dx}[f(x)]^n = n[f(x)]^{n-1}.\frac{d}{dx} f(x)$

7. $\frac{d}{dx} x^n = n\, x^{n-1},\ n \in Q, x \neq 0$

Example:

Find $f'(x)$ for each of the following functions:

1- $f(x) = x^2$ 2- $f(x) = \frac{x+1}{x}$ 3- $f(x) = \sqrt{x+2}$

Sol/

1- $f(x) = x^2 \Rightarrow f'(x) = 2x^{2-1} = 2x$

2- $f(x) = \frac{x+1}{x} \Rightarrow f'(x) = \frac{x(1) - [(x+1)(1)]}{x^2} = \frac{x - x - 1}{x^2} = \frac{-1}{x^2}$

3- $f(x) = \sqrt{x+2} \Rightarrow f'(x) = (\frac{1}{2})(x+2)^{\frac{1}{2}-1}(1) == (\frac{1}{2})(x+2)^{\frac{-1}{2}}$

$$= \frac{1}{2}\frac{1}{\sqrt{x+2}} = \frac{1}{2\sqrt{x+2}}$$

Homework:

Find $f'(x)$ for each of the following functions:

1- $f(x) = x^{-5}$
2- $f(x) = x^{0.7}$
3- $f(x) = (x^2 + 1)(x + 1)$
4- $f(x) = 3x^4(x^3 - 2x)$
5- $f(x) = (3x^3 + 6x)(3x-5)$
6- $f(x) = \frac{(x^2-2x^3)}{x^3-4}$
7- $f(x) = (4x^3 + 3x)^2$