Homogeneous equation

A function $f(x, y)$ is called Homogeneous of degree (n) if $f(\lambda x, \lambda y) = \lambda^n f(x, y)$ for example

1- $f(x, y) = 2x^2 - 3xy - 6y^2$
$f(\lambda x, \lambda y) = 2(\lambda x)^2 - 3(\lambda x)(\lambda y) - 6(\lambda y)^2$
$= \lambda^2(2x^2 - 3xy - 6y^2)$

$\therefore f(x, y)$ is a Homogeneous function of degree (2)

2- $f(x, y) = x^4 + 2x^2y^2$
$f(\lambda x, \lambda y) = (\lambda x)^4 + 2(\lambda x)^2(\lambda y)^2$
$= \lambda^4 x^4 + 2\lambda^4 x^2 y^2$

$\therefore f(x, y)$ is a Homogeneous function of degree (4)

*The first order – first degree ODE can be put in $\frac{dy}{dx} = \frac{f1(x,y)}{f2(x,y)}$ ... ... (1) both $f1(x, y), f2(x, y)$ Are Homogeneous of the same degree, the ODE is said to be Homogeneous let $y = vx \rightarrow \frac{dy}{dx} = v + x\frac{dv}{dx}$ substitution in (1) give $v + x\frac{dv}{dx} = f(v)$ and $x\frac{dv}{dx} = f(v) - v \rightarrow \frac{dv}{f(v)-v} = \frac{dx}{x}$ ... ... (2)

Integration both side of (2) we get general solution.

Examples: find the general solution of the ODE

1- $(2xy - x^2)dy = 2y^2dx \rightarrow \frac{dy}{dx} = \frac{2y^2}{2xy-x^2}$

$\frac{dy}{dx} = \frac{2(\lambda y)^2}{2(\lambda x)(\lambda y)-(\lambda x)^2} \rightarrow \frac{dy}{dx} = \frac{2\lambda^2 y^2}{\lambda^2(2xy-x^2)}$ is Homogeneous

let $y = vx \rightarrow \frac{dy}{dx} = v + x\frac{dv}{dx}$

$v + x\frac{dv}{dx} = \frac{2v^2x^2}{2vx^2-x^2} \rightarrow v + x\frac{dv}{dx} = \frac{v^2 2x^2}{x^2(2v-1)}$

$x\frac{dv}{dx} = \frac{2v^2}{(2v-1)} - v \rightarrow x\frac{dv}{dx} = \frac{2v^2-2v^2+v}{(2v-1)}$

$\int \frac{(2v-1)}{v}dv = \int \frac{dx}{x} \rightarrow \int \frac{2v}{v} - \frac{1}{v}dv = \int \frac{dx}{x}$

$2v - lnv = lnx + c \rightarrow 2(\frac{y}{x}) - \ln(\frac{y}{x}) = lnx + c$

2- $[2x\sinh\left(\frac{y}{x}\right) + 3y\cosh\left(\frac{y}{x}\right)]\, dx = 3x\cosh\left(\frac{y}{x}\right) dy$

$$\frac{dy}{dx} = \frac{2x\sinh\left(\frac{y}{x}\right)+3y\cosh\left(\frac{y}{x}\right)}{3x\cosh\left(\frac{y}{x}\right)} \quad \rightarrow \frac{dy}{dx} = \frac{2(\lambda x)\sinh\left(\frac{\lambda y}{\lambda x}\right)+3(\lambda y)\cosh\left(\frac{\lambda y}{\lambda x}\right)}{3(\lambda x)\cosh\left(\frac{\lambda y}{\lambda x}\right)}$$

$$\frac{dy}{dx} = \frac{\lambda[2x\sinh\left(\frac{y}{x}\right)+3y\cosh\left(\frac{y}{x}\right)]}{\lambda[3x\cosh\left(\frac{y}{x}\right)]}$$ is Homogeneous

let $y = vx \rightarrow \frac{dy}{dx} = v + x\frac{dv}{dx}$

$$v + x\frac{dv}{dx} = \frac{2x\sinh\left(\frac{vx}{x}\right)+3(vx)\cosh\left(\frac{vx}{x}\right)}{3x\cosh\left(\frac{vx}{x}\right)}$$

$$v + x\frac{dv}{dx} = \frac{x[2\sinh(v)+3v\cosh(v)]}{3x\cosh(v)}$$

$$x\frac{dv}{dx} = \frac{2\sinh(v)+3v\cosh(v)}{3\cosh(v)} - v$$

$$x\frac{dv}{dx} = \frac{2\sinh(v)+3v\cosh(v)-3v\cosh(v)}{3\cosh(v)}$$

$$x\frac{dv}{dx} = \frac{2\sinh(v)}{3\cosh(v)} \quad \rightarrow \int\frac{\cosh(v)}{\sinh(v)}dv = \int\frac{2dx}{3x}$$

$$ln|\sinh(v)| = \frac{2}{3}ln|x| + c$$

3- $(x + ylny - ylnx)dx - x(lny - lnx)dy = 0$

$$(x + ylny - ylnx)dx = (xlny - xlnx)dy$$

$$\left(x + y\ln\left(\frac{y}{x}\right)\right)dx = \left(x\ln\left(\frac{y}{x}\right)\right)dy \quad \rightarrow \frac{dy}{dx} = \frac{x+yln\left(\frac{y}{x}\right)}{xln\left(\frac{y}{x}\right)}$$

$$\frac{dy}{dx} = \frac{\lambda x+\lambda yln\left(\frac{\lambda y}{\lambda x}\right)}{\lambda xln\left(\frac{\lambda y}{\lambda x}\right)} \quad \rightarrow \frac{dy}{dx} = \frac{\lambda[x+yln\left(\frac{y}{x}\right)]}{\lambda[xln\left(\frac{y}{x}\right)]}$$ is Homogeneous

let $y = vx \rightarrow \frac{dy}{dx} = v + x\frac{dv}{dx}$

$$v + x\frac{dv}{dx} = \frac{x[1+vln(v)\}}{xln(v)} \quad \rightarrow x\frac{dv}{dx} = \frac{1+vln(v)}{ln(v)} - v$$

$$x\frac{dv}{dx} = \frac{1+vln(v)-vln(v)}{ln(v)} \rightarrow \quad x\frac{dv}{dx} = \frac{1}{ln(v)}$$

$$\int lnv = \int\frac{dx}{x} \rightarrow vlnv - v = lnx + c$$

4- $\left(1+2e^{\frac{x}{y}}\right)dx+2e^{\frac{x}{y}}\left(1-\frac{x}{y}\right)dy=0$

$\left(1+2e^{\frac{x}{y}}\right)dx-2e^{\frac{x}{y}}\left(1-\frac{x}{y}\right)dy$

$\frac{dx}{dy}=\frac{-2e^{\frac{x}{y}}\left(1-\frac{x}{y}\right)}{\left(1+2e^{\frac{x}{y}}\right)} \rightarrow \frac{dx}{dy}=\frac{-2e^{\frac{\lambda x}{\lambda y}}\left(1-\frac{\lambda x}{\lambda y}\right)}{\left(1+2e^{\frac{\lambda x}{\lambda y}}\right)}$ is Homogeneous

let $x=vx \rightarrow \frac{dx}{dy}=v+y\frac{dv}{dy}$

$v+y\frac{dv}{dy}=\frac{2e^{v}(-1+v)}{1+2e^{v}} \rightarrow y\frac{dv}{dy}=\frac{2e^{v}(-1+v)}{1+2e^{v}}-v$

$y\frac{dv}{dy}=\frac{-2e^{v}+2ve^{v}-v-2ve^{v}}{1+2e^{v}} \rightarrow y\frac{dv}{dy}=\frac{-2e^{v}-v}{1+2e^{v}}$

$\int\frac{1+2e^{v}}{2e^{v}+v}dv=\int-\frac{dy}{y} \rightarrow ln|v+2e^{v}|=-ln|y|+c$

<u>Equation that reduces to a Homogeneous Equation</u>

Examples: find the general solution of the ODE

1- $(x-2y-3)dx+(2x+y-1)dy=0$

$x-2y-3=0 \quad \rightarrow -2x+4y+6=0$

$2x+y-1=0 \quad \rightarrow \quad 2x+y-1=0$

$5y=5 \rightarrow y=-1 \rightarrow x=1$

The intersection point (1, -1)   h=1, k=-1

Let $x=x_0+h \quad \rightarrow x=x_0+1 \rightarrow dx=dx_0$

And $y=y_0+k \quad \rightarrow y=y_0-1 \rightarrow dy=dy_0$

$\frac{dy}{dx}=\frac{-x+2y+3}{2x+y-1} \qquad \rightarrow \frac{dy_0}{dx_0}=\frac{-x_0-1+2y_0-2+3}{2x_0+2+y_0-1-1}$

$\frac{dy_0}{dx_0}=\frac{-x_0+2y_0}{2x_0+y_0}$ is Homogeneous

let $y_0=v_0x_0 \quad \rightarrow \frac{dy_0}{dx_0}=v_0+x_0\frac{dv_0}{dx_0}$

$v_0+x_0\frac{dv_0}{dx_0}=\frac{-x_0+2v_0x_0}{2x_0+v_0x_0} \qquad \rightarrow v_0+x_0\frac{dv_0}{dx_0}=\frac{-1+2v_0}{2+v_0}$

$x_0\frac{dv_0}{dx_0}=\frac{-1+2v_0}{2+v_0}-v_0 \qquad \rightarrow x_0\frac{dv_0}{dx_0}=\frac{-1+2v_0-2v_0-v_0^2}{2+v_0}$

$\int \frac{2+v_0}{-1-v_0^2} dv_0 = \int \frac{dx_0}{x_0} \rightarrow \int \frac{2}{-1-v_0^2} dv_0 + \int \frac{v_0}{-1-v_0^2} dv_0 = \int \frac{dx_0}{x_0}$

$-2\tan^{-1} v_0 - \frac{1}{2} ln|1 + v_0^2| = ln|x_0| + c$

$-2\tan^{-1}(\frac{y_0}{x_0}) - \frac{1}{2} ln\left|1 + (\frac{y_0}{x_0})^2\right| = ln|x_0| + c$

$-2\tan^{-1}(\frac{y+1}{x-1}) - \frac{1}{2} ln\left|1 + (\frac{y+1}{x-1})^2\right| = ln|x_0| + c$

2- $(x - y - 1)dx + (4y + x - 1)dy = 0$

$x - y - 1 = 0 \quad \rightarrow \quad y - x + 1 = 0$

$\underline{4y + x - 1 = 0 \quad \rightarrow \quad 4y + x - 1 = 0}$

$5y = 0 \rightarrow y = 0 \rightarrow x = 1$

The intersection point (1, 0) h= 1, k= 0

Let $x = x_0 + h \quad \rightarrow x = x_0 + 1 \rightarrow dx = dx_0$

And $y = y_0 + k \quad \rightarrow y = y_0 + 0 \rightarrow dy = dy_0$

$\frac{dy}{dx} = \frac{y-x+1}{4y+x-1} \quad \rightarrow \quad \frac{dy_0}{dx_0} = \frac{y_0-x_0-1+1}{4y_0+x_0+1-1}$ is Homogeneous

let $y_0 = v_0 x_0 \quad \rightarrow \quad \frac{dy_0}{dx_0} = v_0 + x_0 \frac{dv_0}{dx_0}$

$v_0 + x_0 \frac{dv_0}{dx_0} = \frac{v_0 x_0 - x_0}{4v_0 x_0 + x_0} \quad \rightarrow v_0 + x_0 \frac{dv_0}{dx_0} = \frac{v_0-1}{4v_0+1}$

$x_0 \frac{dv_0}{dx_0} = \frac{v_0-1}{4v_0+1} - v_0 \rightarrow x_0 \frac{dv_0}{dx_0} = \frac{v_0-1-4v_0^2-v_0}{4v_0+1}$

$x_0 \frac{dv_0}{dx_0} = \frac{-1-4v_0^2}{4v_0+1} \rightarrow \; -\int \frac{4v_0+1}{1+4v_0^2} dv_0 = \int \frac{dx_0}{x_0}$

$\int \frac{4v_0}{1+4v_0^2} dv_0 + \int \frac{1}{1+4v_0^2} = -\int \frac{dx_0}{x_0}$

$\frac{1}{2} ln|1 + 4v_0^2| + \tan^{-1} 2v_0 = -ln|x_0| + c$

$\frac{1}{2} ln\left|1 + 4(\frac{y_0}{x_0})^2\right| + \tan^{-1} 2(\frac{y_0}{x_0}) = -ln|x_0| + c$

$\frac{1}{2} ln\left|1 + 4(\frac{y}{x-1})^2\right| + \tan^{-1} 2(\frac{y}{x-1}) = -ln|x - 1| + c$

H.W. Determine Homogeneous or non-Homogeneous

1) $f(x) = y + \sqrt{x^2 + xy}$

2) $f(x) = x + y \sin(\frac{y}{x})$

3) $f(x) = \frac{\ln(x^2 - 2xy + y^2)}{x^2 + y^2}$

4) $f(x) = 1 + \tan\frac{y^2}{x}$

H.W. Find the general solution of the differential equation

1) $x^2 y' = xy + x^2 + y^2$

2) $y' = \frac{2y^2 + x^2 e^{-\left(\frac{y}{x}\right)^2}}{2xy}$

3) $y' = \frac{y^3 + 2xy^2 + x^2 y + x^3}{x(y+x)^2}$

4) $\frac{dy}{dx} = \frac{xy^3}{2y^4 + x^4}$

5) $\left[xyln\left(\frac{x}{y}\right)\right]dx + [y^2 - x^2 ln\left(\frac{x}{y}\right)]dy = 0$

6) $(6x - y + 3)dx + (2x - y - 1)dy = 0$

7) $y' = \frac{2x + y + 1}{x + 2y - 4}$

8) $y' = \frac{2xye^{(\frac{x}{y})^2}}{y^2 + y^2 e^{(\frac{x}{y})^2} + 2x^2 e^{(\frac{x}{y})^2}}$

9) $(ydx + xdy)x\cos\frac{y}{x} = (xdy - ydx)y\sin\frac{y}{x}$

10) $x\tan\frac{y}{x} + x\sec^2\frac{y}{x}\frac{dy}{dx} = y\sec^2\frac{y}{x}$