Exact differential equation

The ODE of first order – first degree $M(x,y)dx + N(x,y)dy = 0$.... (1) is said to be exact iff $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$.

If (1) is an exact equation, then there exists a function $f(x,y) = c$ is a general solution of (1) where $dF = M(x,y)dx + N(x,y)dy = 0$

where $M(x,y) = \frac{\partial f}{\partial x}$, $N(x,y) = \frac{\partial f}{\partial y}$ Integrating the above equation, we get $F = c$

Examples: find the general solution of the ODE

1- $(4x^3y^3 + 3x^2)dx + (3x^4y^2 + 6y^2)dy = 0$

$M = 4x^3y^3 + 3x^2 \rightarrow \frac{\partial M}{\partial y} = 12x^3y^2$

$N = 3x^4y^2 + 6y^2 \rightarrow \frac{\partial N}{\partial x} = 12x^3x^2$

$\because \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ Then the ODE is exact and $F = c$ is the general solution.

$F = \int M \ dx$ then $\frac{\partial f}{\partial y} = N$ Or $F = \int N \ dy$ then $\frac{\partial f}{\partial x} = M$

$F = \int M \ dx \rightarrow F = \int (4x^3y^3 + 3x^2) \, dx$

$F = x^4y^3 + x^3 + h(y)$

$\frac{\partial f}{\partial y} = 3x^4y^2 + h'(y) = 3x^4y^2 + 6y^2$

$h'(y) = 6y^2 \rightarrow h(y) = 2y^3$

The general solution is $x^4y^3 + x^3 + 2y^3 = c$

2- $(3y\cos x + 4xe^x + 2x^2e^x)dx + (3\sin x + 3)dy = 0$

$M = 3y\cos x + 4xe^x + 2x^2e^x \;\rightarrow\; \frac{\partial M}{\partial y} = 3\cos x$

$N = 3\sin x + 3 \;\rightarrow\; \frac{\partial N}{\partial x} = 3\cos x$

$\because \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ Then the ODE is exact and $F = c$ is the general solution.

$F = \int M\ dx$ then $\frac{\partial f}{\partial y} = N$ Or $F = \int N\ dy$ then $\frac{\partial f}{\partial x} = M$

$F = \int N\ dy \;\rightarrow\; F = \int (3\sin x + 3)\,dy$

$F = 3y\sin x + 3y + h(x)$

$\frac{\partial f}{\partial x} = 3y\cos x + h'(x) = 3y\cos x + 4xe^x + 2x^2e^x$

$h'(x) = 4xe^x + 2x^2e^x$

$h(x) = 4(xe^x - e^x) + 2(x^2e^x - 2xe^x + 2e^x)$
$h(x) = 4xe^x - 4e^x + 2x^2e^x - 4xe^x + 4e^x$
$h(x) = 2x^2e^x$
The general solution is $3y\sin x + 3y + 2x^2e^x = c$

3- $\left[y\left(1 - \frac{1}{x}\right) + \cos y\right]dx = [x\sin y + lnx - x]dy$
$\left[y\left(1 - \frac{1}{x}\right) + \cos y\right]dx + [-x\sin y - lnx + x]dy = 0$

$M = y\left(1 - \frac{1}{x}\right) + \cos y \;\rightarrow\; \frac{\partial M}{\partial y} = 1 - \frac{1}{x} - \sin y$

$N = -x\sin y - lnx + x \rightarrow \frac{\partial N}{\partial x} = -\sin y - \frac{1}{x} = 1$

$\because \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ Then the ODE is exact and $F = c$ is the general solution

$F = \int M\ dx$ then $\frac{\partial f}{\partial y} = N$ Or $F = \int N\ dy$ then $\frac{\partial f}{\partial x} = M$

$F = \int M \; dx \;\; \rightarrow \;\; F = \int y\left(1 - \frac{1}{x}\right) + \cos y \; dx$

$F = yx - ylnx + +x\cos y + h(y)$

$\frac{\partial f}{\partial y} = x - lnx - x\sin y + h'(y) = -x\sin y - lnx + x$

$h'(y) = 0 \;\; \rightarrow \; h(y) = c$

The general solution is $F = yx - ylnx + +x\cos y + c$

Not an exact differential equation

- **Integration factor I.F**: if $M(x,y)dx + N(x,y)dy = 0 \ldots\ldots\ldots (1)$

  If I(x) is a function of x only, where

  $f(x) = \frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} \quad \rightarrow \quad I(x) = e^{\int f(x)dx}$ Is the integration factor

  If I(y) is a function of y only, where

  $f(y) = \frac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{M} \quad \rightarrow \quad I(y) = e^{\int f(y)dy}$

Since If I is the integration factor of (1) then

$I.M(x,y)dx + I.N(x,y)dy = 0 \;\; \rightarrow \;\; \frac{\partial I.M}{\partial y} = \frac{\partial I.N}{\partial x}$

Examples: find the general solution of the ODE

1- $(2xy^3 - 2x^3y^3 - 4xy^2 + 2x)dx + (3x^2y^2 + 4y)dy = 0$

$M = 2xy^3 - 2x^3y^3 - 4xy^2 + 2x \;\; \rightarrow \;\; \frac{\partial M}{\partial y} = 6xy^2 - 6x^3y^2 - 8xy$

$N = 3x^2y^2 + 4y \;\; \rightarrow \; \frac{\partial N}{\partial x} = \; 6xy^2$

$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$ not exact

$$f(x)=\frac{\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}}{N} \quad \rightarrow \; f(x)=\frac{6xy^2-6x^3y^2-8xy-6xy^2}{3x^2y^2+4y}$$

$$f(x)=\frac{-2x(3x^2y^2+4y)}{3x^2y^2+4y} \quad \rightarrow \quad f(x)=-2x$$

Since $I(x)=e^{\int f(x)dx} \;\rightarrow\; I(x)=e^{\int -2x\,dx} \;\rightarrow\; I(x)=e^{-x^2}$

$$[(2xy^3-2x^3y^3-4xy^2+2x)dx+(3x^2y^2+4y)dy=0]\;\left(e^{-x^2}\right)$$

$$\left(2xy^3e^{-x^2}-2x^3y^3e^{-x^2}-4xy^2e^{-x^2}+2xe^{-x^2}\right)dx+$$
$$\left(3x^2y^2e^{-x^2}+4ye^{-x^2}\right)dy=0$$

$$\frac{\partial I.M}{\partial y}=6xy^2e^{-x^2}-6x^3y^2e^{-x^2}-8xye^{-x^2}$$

$$\frac{\partial I.N}{\partial x}=6xy^2e^{-x^2}-6x^3y^2e^{-x^2}-8xye^{-x^2}$$

$\frac{\partial I.M}{\partial y}=\frac{\partial I.N}{\partial x}$ Then the ODE is exact and $F=c$ is the general solution

$F=\int M\;dx$ then $\frac{\partial f}{\partial y}=N$ Or $F=\int N\;dy$ then $\frac{\partial f}{\partial x}=M$

$$F=\int N\;dy \;\rightarrow\; F=\int 3x^2y^2e^{-x^2}+4ye^{-x^2}\;dy$$

$$F=x^2y^3e^{-x^2}+2y^2e^{-x^2}+h(x)$$

$$\frac{\partial f}{\partial x}=-2x^3y^3e^{-x^2}+2xy^3e^{-x^2}-4xy^2e^{-x^2}+h'(x)=2xy^3e^{-x^2}-$$
$$2x^3y^3e^{-x^2}-4xy^2e^{-x^2}+2xe^{-x^2}$$

$$h'(x)=2xe^{-x^2} \quad \rightarrow \; h(x)=-e^{-x^2}$$

The general solution is $F=x^2y^3e^{-x^2}+2y^2e^{-x^2}-e^{-x^2}=c$

2- $(2xy^4e^y + 2xy^3 + y)dx + (x^2y^4e^y - x^2y^2 - 3x)dy = 0$

$M = 2xy^4e^y + 2xy^3 + y \rightarrow \quad \frac{\partial M}{\partial y} = 2xy^4e^y + 8xy^3e^y + 6xy^2 + 1$

$N = x^2y^4e^y - x^2y^2 - 3x \quad \rightarrow \frac{\partial N}{\partial x} = \ 2xy^4e^y - 2xy^2 - 3$

$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$ not exact

$f(y) = \frac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{M} \quad \rightarrow f(y) = \frac{2xy^4e^y - 2xy^2 - 3 - 2xy^4e^y - 8xy^3e^y - 6xy^2 - 1}{2xy^4e^y + 2xy^3 + y}$

$f(y) = \frac{-8xy^2 - 8xy^3e^y - 4}{2xy^4e^y + 2xy^3 + y} \rightarrow f(y) = \frac{-4(2xy^2 + 2xy^3e^y + 1)}{y(2xy^3e^y + 2xy^2 + 1)}$

$f(y) = \frac{-4}{y} \quad \rightarrow \quad I(y) = e^{\int f(y)dy} \quad \rightarrow I(y) = e^{\int \frac{-4}{y}dy}$

$I(y) = e^{-4lny} \quad \rightarrow I(y) = e^{lny^{-4}} \quad \rightarrow I(y) = y^{-4} \quad \rightarrow \ I(y) = \frac{1}{y^4}$

$[(2xy^4e^y + 2xy^3 + y)dx + (x^2y^4e^y - x^2y^2 - 3x)dy = 0](\frac{1}{y^4})$

$\left(2xe^y + \frac{2x}{y} + \frac{1}{y^3}\right) dx + \left(x^2e^y - \frac{x^2}{y^2} - \frac{3x}{y^4}\right) dy = 0$

$\frac{\partial I.M}{\partial y} = 2xe^y - \frac{x^2}{y^2} - \frac{3}{y^4}$

$\frac{\partial I.N}{\partial x} = 2xe^y - \frac{x^2}{y^2} - \frac{3}{y^4}$

$\frac{\partial I.M}{\partial y} = \frac{\partial I.N}{\partial x}$ Then the ODE is exact and $F = c$ is the general solution

$F = \int M \ dx$ then $\frac{\partial f}{\partial y} = N$ Or $F = \int N \ dy$ then $\frac{\partial f}{\partial x} = M$

$F = \int M \ dx \rightarrow F = \int 2xe^y + \frac{2x}{y} + \frac{1}{y^3} \ dx$

$F = x^2e^y + \frac{x^2}{y} + \frac{x}{y^3} + h(y)$

$$\frac{\partial f}{\partial x} = x^2 e^y - \frac{x^2}{y^2} - \frac{3x}{y^4} + h'(y) = x^2 e^y - \frac{x^2}{y^2} - \frac{3x}{y^4}$$

$h'(y) = 0 \;\; \rightarrow \;\; h(y) = a$

The general solution is $F = x^2 e^y + \frac{x^2}{y} + \frac{x}{y^3} + a$

❖ **Alternative method:** The following equation are exact

1- $xdy + ydx = 0$ can be written as $d(x.y)$
2- $xdx + ydy = 0$ can be written as $\frac{1}{2}d(x^2 + y^2)$
3- $\frac{xdy - ydx}{x^2} = 0$ can be written as $d(\frac{y}{x})$
4- $\frac{ydx - xdy}{y^2} = 0$ can be written as $d(\frac{x}{y})$
5- $\frac{xdy + ydx}{xy} = 0$ can be written as $d[\ln(x.y)]$
6- $\frac{xdy - ydx}{xy} = 0$ can be written as $d[\ln(\frac{y}{x})]$
7- $\frac{ydx - xdy}{xy} = 0$ can be written as $d[\ln(\frac{x}{y})]$

Examples: find the general solution of the ODE

1- $(3x + 5)dx - 2(ydx + xdy) + (4y + 1)dy = 0$

$(3x + 5)dx - 2d(x.y) + (4y + 1)dy = 0$

$\int(3x + 5)dx - 2\int d(x.y) + \int(4y + 1)dy = 0$

$\frac{3x^2}{2} + 5x - 2xy + 2y^2 + y = c$

2- $xdx + \frac{ye^x dx + e^x dy}{y^2} = 0$

$\int d\left(\frac{x^2}{2}\right) + \int d(\frac{e^x}{y})$

$\frac{x^2}{2} + \frac{e^x}{y} = c$

3- $[ydx - xdy + 2x^2 y \sin x^2\ dx = 0] \div (xy)$

$\frac{ydx-xdy}{xy} + 2x \sin x^2\ dx = 0$

$\int d[\ln(\frac{x}{y})] + \int 2x \sin x^2\ dx = 0$

$\ln(\frac{x}{y}) - \cos x^2 = 0$

H.W. Determine the following equation are exact or not

1 - $(2x\cos y + 3x^2 y)dx + (x^3 - x^2 \sin y - y)dy = 0$

2 - $(x^2 e^{x+y} + ye^{x+y} + 2x)dx + (x^2 e^{x+y} + 4y)dy = 0$

3 - $\ln(y^2+1)\, dx + \frac{2y(x-1)}{y^2+1} dy = 0$

H.W. Find the general solution of the differential equation

1 – $\left(e^x + lny + \frac{y}{x}\right) dx + \left(\frac{x}{y} + lnx + \sin y\right) dy = 0$

2 – $\left(\frac{y^2}{1+x^2} - 2y\right) dx + (2y \tan^{-1} x - 2x + \sinh y)dy = 0$

3 - $\left(x + \sqrt{y^2+1}\right)dx - \left(y - \frac{xy}{\sqrt{y^2+1}}\right) dy = 0$

4 - $\frac{dy}{dx} = \frac{4-3x^2}{x^3}$

5 - $\left(y^2 e^{xy^2} + 4x^3\right)dx + \left(2xye^{xy^2} - 3y^2\right)dy = 0$

6 - $(y-4)(y+1)dx + (3-2y)(x+1) = 0$

7 - $6dx + (4x - 2y - 1)dy = 0$

8 – $y(y + 3x + 2)dx + x(y + x + 1)dy = 0$

9 - $\frac{xdx+ydy}{x^2+y^2} + \tan^{-1} y\ \ dy = 0$

10 - $2y\cos x\, dy - y^2 \sin x\, dx = \sin x\, dy + y\cos x\, dx$

H.W. Find the I.F for each the following not exact equation

1 – $(2y \sin x + 3y^4 \sin x \cos x)dx - (4y^3 \cos^2 x + \cos x)dy = 0$

2 – $(3x^2 y - x^2)dx + dy = 0$

3 - $2xy^3 dx + (3x^2 y^2 + x^2 y^3 + 1)dy = 0$