<u>Linear differential equation</u>: the equation has form

$\frac{dy}{dx} + P(x).y = Q(x) \dots\dots\dots(1)$

From (1) find $I(x) = e^{\int P(x)dx} \rightarrow I(x).y = \int I(x).Q(x)\,dx$

$\frac{dx}{dy} + P(y).x = Q(y) \dots\dots\dots(2)$

From (2) find $I(y) = e^{\int P(y)dy} \rightarrow I(y).y = \int I(y).Q(y)\,dx$

Examples: find the general solution of the ODE

1- $\frac{dy}{dx} + \frac{y}{x} = 4x^2 \qquad P(x) = \frac{1}{x} \qquad Q(x) = 4x^2$

$I(x) = e^{\int P(x)dx} \rightarrow I(x) = e^{\int \frac{1}{x}\,dx} \rightarrow I(x) = e^{\ln(x)} = x$

$I(x).y = \int I(x).Q(x)\,dx \rightarrow x.y = \int x.4x^2\,dx$

$x.y = \int 4x^3\,dx \rightarrow x.y = x^4 + c$

$y = x^3 + \frac{c}{x}$

2- $\cos x \frac{dy}{dx} + y \sin x = 1$

$[\cos x \frac{dy}{dx} + y \sin x = 1] \div \cos x$

$\frac{dy}{dx} + y \tan x = \sec x \qquad P(x) = \tan x \qquad Q(x) = \sec x$

$I(x) = e^{\int P(x)dx} \rightarrow I(x) = e^{\int \tan x\,dx} \rightarrow I(x) = e^{-\ln \cos x}$

$I(x) = (\cos x)^{-1} \rightarrow I(x) = \frac{1}{\cos x} = \sec x$

$I(x).y = \int I(x).Q(x)\,dx \rightarrow \sec x.y = \int \sec x.\sec x\,dx$

$\sec x.y = \int \sec^2 x\,dx \rightarrow \sec x.y = \tan x + c$

$y = \frac{\tan x}{\sec x} + \frac{c}{\sec x} \rightarrow y = \sin x + c\cos x$

3- $\frac{dy}{dx} + 3y = e^{-3x}$ $\quad P(x) = 3 \quad Q(x) = e^{-3x}$

$I(x) = e^{\int P(x)dx} \rightarrow I(x) = e^{\int 3\ dx} \rightarrow I(x) = e^{3x}$

$I(x).y = \int I(x).Q(x)\ dx \rightarrow e^{3x}.y = \int e^{3x}.e^{-3x}\ dx$

$e^{3x}.y = \int 1\ dx \rightarrow e^{3x}.y = x + c$

$y = \frac{x}{e^{3x}} + \frac{c}{e^{3x}} \rightarrow y = xe^{-3x} + ce^{-3x}$

4- $\frac{dy}{dx} + \frac{y}{x} = \sin x$ $\quad P(x) = \frac{1}{x} \quad Q(x) = \sin x$

$I(x) = e^{\int P(x)dx} \rightarrow I(x) = e^{\int \frac{1}{x}\ dx} \rightarrow I(x) = e^{\ln(x)} = x$

$I(x).y = \int I(x).Q(x)\ dx \rightarrow x.y = \int x.\sin x\ dx$

$x.y = -x\cos x - \sin x + c$

$y = -\cos x - \frac{\sin x}{x} + \frac{c}{x}$

5- $yln y\ dx + (x - ln y)dy = 0$

$[yln y\ dx + (x - ln y)dy = 0] \div dy$

$yln y \frac{dx}{dy} + x - ln y = 0 \rightarrow yln y \frac{dx}{dy} + x = ln y$

$\frac{dx}{dy} + \frac{x}{yln y} = \frac{1}{y} \quad P(y) = \frac{1}{yln y} \quad Q(y) = \frac{1}{y}$

$I(y) = e^{\int P(y)dy} \rightarrow I(y) = e^{\int \frac{1}{yln y}\ dy} \rightarrow I(y) = e^{\ln(\ln y)} = ln y$

$I(y).x = \int I(y).Q(y)\ dy \rightarrow ln y.x = \int ln y.\frac{1}{y}\ dy$

$ln y.x = \frac{(ln y)^2}{2} + c \rightarrow x = \frac{ln y}{2} + \frac{c}{ln y}$

<u>Bernoulli differential equation</u>:

A) the equation has form

$\frac{dy}{dx} + P(x).y = Q(x).y^n \dots\dots\dots(1)$ division of both side of (1) by y^n

$y^{-n}\frac{dy}{dx} + P(x).y^{1-n} = Q(x) \dots\dots\dots..(2)$

Let $z = y^{1-n} \quad \rightarrow \quad \frac{dz}{dx} = (1-n)y^{-n}\frac{dy}{dx}$ substitution (2) we get

$\frac{dz}{dx} + (1-n)P(x).z = Q(x) \dots\dots\dots(3)$

OR

$\frac{dx}{dy} + P(x).x = Q(y).x^n \dots\dots\dots(1)$ division of both side of (1) by x^n

$x^{-n}\frac{dx}{dy} + P(y).x^{1-n} = Q(y) \dots\dots\dots..(2)$

Let $z = x^{1-n} \quad \rightarrow \quad \frac{dz}{dy} = (1-n)x^{-n}\frac{dx}{dy}$ substitution (2) we get

$\frac{dz}{dy} + (1-n)P(y).z = Q(y) \dots\dots\dots(3)$

Examples: find the general solution of the ODE

1- $y(6y^2 - x - 1)dx + 2xdy = 0$

$[(6y^3 - xy - y)dx + 2xdy = 0] \div dx$

$6y^3 - y(x+1) + 2x\frac{dy}{dx} = 0$

$[2x\frac{dy}{dx} - (x+1)y = -6y^3] \div 2x$

$[\frac{dy}{dx} - \frac{(x+1)y}{2x} = \frac{-3}{x}y^3] \div y^3$

$\frac{dy}{dx}y^{-3} - \frac{(x+1)}{2x}y^{-2} = \frac{-3}{x}$

Let $z = y^{-2} \rightarrow \frac{dz}{dx} = -2y^{-3}\frac{dy}{dx} \rightarrow \frac{dy}{dx} = \frac{1}{-2y^{-3}}\frac{dz}{dx}$

$$[\frac{1}{-2}\frac{dz}{dx} - \frac{(x+1)}{2x}z = \frac{-3}{x}](-2)$$

$$\frac{dz}{dx} + \frac{(x+1)}{x}z = \frac{6}{x} \qquad P(x) = \frac{x+1}{x} \qquad Q(x) = \frac{6}{x}$$

$$I(x) = e^{\int P(x)dx} \rightarrow I(x) = e^{\int\frac{x+1}{x}\,dx} \rightarrow I(x) = e^{\text{x}+\ln(x)} = xe^{x}$$

$$I(x).z = \int I(x).Q(x)\,dx \rightarrow xe^{x}.z = \int xe^{x}.\frac{6}{x}\,dx$$

$$xe^{x}.z = 6e^{x} + c \rightarrow z = \frac{6e^{x}+c}{xe^{x}} \rightarrow y^{-2} = \frac{6e^{x}+c}{xe^{x}}$$

2- $ydx = (x + x^{3}y\cos y)dy$

$$[ydx = (x + x^{3}y\cos y)dy] \div dy$$

$$[y\frac{dx}{dy} = (x + x^{3}y\cos y)dy] \div y$$

$$\frac{dx}{dy} = \frac{x}{y} + x^{3}\cos y \rightarrow \left[\frac{dx}{dy} - \frac{x}{y} = x^{3}\cos y\right] \div x^{3}$$

$$x^{-3}\frac{dx}{dy} - \frac{1}{y}x^{-2} = \cos y$$

Let $z = x^{-2} \rightarrow \frac{dz}{dy} = -2x^{-3}\frac{dx}{dy} \rightarrow \frac{dx}{dy} = \frac{1}{-2x^{-3}}\frac{dz}{dy}$

$$\frac{1}{-2}\frac{dz}{dy} - \frac{1}{y}z = \cos y$$

$$\frac{dz}{dy} + \frac{2}{y}z = -2\cos y \qquad P(y) = \frac{2}{y} \qquad Q(y) = -2\cos y$$

$$I(y) = e^{\int P(y)dy} \rightarrow I(y) = e^{\int\frac{2}{y}\,dy} \rightarrow I(y) = e^{2\ln(y)} = y^{2}$$

$$I(y).z = \int I(y).Q(y)\,dy \rightarrow y^{2}.z = \int y^{2}.-2\cos y\;dy$$

$$y^{2}.z = -2\sin y - 4y\cos y + 4\sin y + c$$

$$z = \frac{-2\sin y - 4y\cos y + 4\sin y + c}{y^{2}}$$

$$x^{-2} = \frac{-2\sin y - 4y\cos y + 4\sin y + c}{y^{2}}$$

B) the equation has form

$$f'(y)\frac{dy}{dx} + P(x).f(y) = Q(x) \;\ldots\ldots\ldots\ldots\ldots.(1)$$

Let $z = f(y) \rightarrow \frac{dz}{dx} = f'(y)\frac{dy}{dx}$

$$\frac{dz}{dx} + P(x).z = Q(x) \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots.(2)$$

OR

$$f'(x)\frac{dx}{dy} + P(y).f(x) = Q(y) \;\ldots\ldots\ldots\ldots\ldots.(1)$$

Let $z = f(x) \rightarrow \frac{dz}{dy} = f'(x)\frac{dx}{dy}$

$$\frac{dz}{dy} + P(y).z = Q(y) \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots.(2)$$

Examples: find the general solution of the ODE

1- $\sec^2 y\frac{dy}{dx} + 2x\tan y = x^3$

Let $z = \tan y \rightarrow \frac{dz}{dx} = \sec^2 y\frac{dy}{dx}$

$\frac{dz}{dx} = 2x.z = x^2 \qquad P(x) = 2x \qquad Q(x) = x^3$

$I(x) = e^{\int P(x)dx} \rightarrow I(x) = e^{\int 2x\,dx} \rightarrow I(x) = e^{x^2}$

$I(x).z = \int I(x).Q(x)\,dx \rightarrow e^{x^2}.z = \int e^{x^2}.x^3\,dx$

Let $u = x^2 \rightarrow du = 2x\,dx \rightarrow dx = \frac{1}{2x}\,du$

$e^{x^2}.z = \int e^{u}.u.x.\frac{1}{2x}\,du \rightarrow e^{x^2}.z = \frac{1}{2}\int e^{u}.u\;du$

$e^{x^2}.z = \frac{1}{2}[\,ue^{u} - e^{u}] + c$

$e^{x^2}.z = \frac{1}{2}\left[x^2e^{x^2} - e^{x^2}\right] + c$

$z = \frac{1}{2}[x^2 - 1] + c$

$\tan y = \frac{1}{2}[x^2 - 1] + c$

2- $2xy\, dy + (1 + y^2)\, dx = (x - 2)e^x dx$

$[2xy\, dy + (1 + y^2)\, dx = (x - 2)e^x dx] \div dx$

$[2xy \frac{dy}{dx} + (1 + y^2) = (x - 2)e^x] \div (x)$

$2y \frac{dy}{dx} + \frac{1}{x}(1 + y^2) = \frac{(x-2)}{x} e^x$

Let $z = 1 + y^2 \rightarrow \frac{dz}{dx} = 2y \frac{dy}{dx}$

$\frac{dz}{dx} + \frac{1}{x}.z = \frac{(x-2)}{x} e^x \qquad P(x) = \frac{1}{x} \qquad Q(x) = \frac{(x-2)}{x} e^x$

$I(x) = e^{\int P(x)dx} \rightarrow I(x) = e^{\int \frac{1}{x} dx} \rightarrow I(x) = e^{lnx} = x$

$I(x).z = \int I(x).Q(x)\, dx \rightarrow x.z = \int x.\frac{(x-2)}{x} e^x\, dx$

$x.z = \int (x - 2)e^x\, dx \rightarrow x.z = \int (x - 2)e^x\, dx$

$x.z = (x - 2)e^x - e^x + c$

$z = \frac{(x-2)}{x} e^x - \frac{1}{x} e^x + \frac{c}{x}$

$1 + y^2 = \frac{(x-2)}{x} e^x - \frac{1}{x} e^x + \frac{c}{x}$

H.W. Find the general solution of the differential equation

1-- $y' + y \cot x = \csc x$ 2-- $\left[(1 + y^2) - \frac{x}{y}\right] \frac{dy}{dx} = 1$

3-- $x\, dy - [y + xy^3(1 + lnx)]dx = 0$

4--$(x + 2)^2 y' = 5 - 8y - 4xy$

5-- $\cos^2 x \sin x\ dy + (y \cos^3 x - 1)dx = 0$

6-- $(x^2 - y^4) \frac{dy}{dx} = xy$ 7--$2x^2(x + 1)y' - \frac{2x}{y} y' = 1$

8-- $(\ln y)y' + y(lny)^2 = y \sinh 3x$

9-- $y - (\cos x)y' = y^2 \cos x\, (1 - \sin^2 x)$

10-- $\frac{x}{\sqrt{1+y^2}} y' + \sinh^{-1} y = x^2$