<u>Application of ODE (Newton's law of cooling)</u>

Newton's law of cooling expresses through the following first order – first degree ODE.

$\frac{dT}{dt} = -k(T - T_0)$

$\frac{dT}{T-T_0} = -k\ dt$ integrating both sides

$ln|T - T_0| = -kt + c$

$e^{ln|T-T_0|} = e^{-kt+c}$

$T - T_0 = ae^{-kt}$ $where$ $a = e^c$

$T = T_0 + ae^{-kt}$

EX1: The temperature of body temperature initially at 80 c^o reduces 60 c^o In 12 min if the temperature of the surrounding air is 30 c^o Find the temperature of the body after 24 min.

Sol:

$\frac{dT}{dt} = -k(T - T_0)$

$\frac{dT}{T-T_0} = -k\ dt$ integrating both sides

$ln|T - T_0| = -kt + c$

$e^{ln|T-T_0|} = e^{-kt+c}$

$T - T_0 = ae^{-kt}$ $where$ $a = e^c$

$T = T_0 + ae^{-kt}$

$80 = 30 + ae^{-0k}$ $\qquad T = 80\ \ T_0 = 30\ \ t = 0$

$a = 50$

$60 = 30 + 50e^{-12k} \quad \rightarrow \quad 30 = 50e^{-12k} \qquad T = 60 \quad T_0 = 30 \quad t = 12$

$e^{-12k} = \frac{3}{5} \quad \rightarrow \quad -12k = \ln\left(\frac{3}{5}\right)$

$k = \frac{1}{12} ln \left(\frac{3}{5}\right)^{-1} \quad \rightarrow \quad k = \frac{1}{12} ln \frac{5}{3}$

$T = 30 + 50e^{-24\left(\frac{1}{12} ln\frac{5}{3}\right)} \qquad T = ? \quad T_0 = 30 \quad t = 24$

$T = 30 + 50\left(\frac{3}{5}\right)^2 \quad \rightarrow \quad T = 30 + 50\left(\frac{9}{25}\right)$

$T = 48$

EX2: body is heated to 150 c^o and placed in air at 15 c^o After 1 hr the temperature became 60 c^o How many additional times is required for it to cool to 30 c^o

Sol:

$\frac{dT}{dt} = -k(T - T_0)$

$\frac{dT}{T-T_0} = -k\ dt$ integrating both sides

$ln|T - T_0| = -kt + c$

$e^{ln|T-T_0|} = e^{-kt+c}$

$T - T_0 = ae^{-kt} \quad where \qquad a = e^c$

$T = T_0 + ae^{-kt}$

$150 = 15 + ae^{-0k} \qquad T = 150 \quad T_0 = 15 \quad t = 0$

$a = 135$

$60 = 15 + 135e^{-(1)k} \rightarrow \quad 45 = 135e^{-k} \qquad T = 60 \quad T_0 = 15 \quad t = 1$

$e^{-k} = \frac{1}{3} \quad \rightarrow \quad -k = \ln\left(\frac{1}{3}\right)$

$k = ln\,(\frac{1}{3})^{-1} \;\rightarrow\; k = ln3$

$30 = 15 + 135e^{-(ln3)t}$ $\qquad T = 30 \quad T_0 = 15 \quad t = ?$

$15 = 135(\frac{1}{3})^t \;\rightarrow\; \frac{15}{135} = (\frac{1}{3})^t$

$\frac{1}{9} = (\frac{1}{3})^t \;\rightarrow\; (\frac{1}{3})^2 = (\frac{1}{3})^t$

$t = 2 \;\rightarrow\; hr = 2 - 1 = 1$

H.W1: If the temperature of the air is $20\ c^o$ a body cool from $140\ c^o$ to $80\ c^o$ In $20\ min$, how much time will it take to reduce to $35\ c^o$

H.W2: a body initially at $80\ c^o$ Cool down to $60\ c^o$ In $20\ min$ the temperature of the air is $40\ c^o$ Find the temperature of the body after $40\ min$

1.5. Differential equation of first order and higher degree

Let $p = \frac{dy}{dx}, p^2 = \frac{dy^2}{dx}, \ldots\ldots\ldots\ldots\ldots\, p^n = \frac{dy^n}{dx}$

1.5.1. Equation solvable for (p)

The ODE it can be resolved in to (n) linear factor in (p) of the type

$[p - f_1(x,y)][p - f_2(x,y)] \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots [p - f_n(x,y)] = 0$

We can equate each factor to zero and resulting DE of 1-st order and 1-st degree can be solved then the solution

$\phi(x,y,c_1) = 0\,, \phi(x,y,c_2) = 0\,, \ldots\ldots\ldots\ldots\ldots\ldots \phi(x,y,c_n) = 0$

Examples: find the general solution of the ODE

1- $p^2 + 4p + 3 = 0 \quad \rightarrow \quad (p+3)(p+1) = 0$

$p + 3 = 0 \quad \rightarrow \quad \frac{dy}{dx} + 3 = 0 \quad \rightarrow \quad \frac{dy}{dx} = -3$

$dy = -3dx \quad \rightarrow \quad y = -3x + c1$

$p + 1 = 0 \quad \rightarrow \quad \frac{dy}{dx} + 1 = 0 \quad \rightarrow \quad \frac{dy}{dx} = -1$

$dy = -dx \quad \rightarrow \quad y = -x + c2$

The general solution is $(y + 3x - c1)(y + x - c2\ ) = 0$

2- $p^2 - 2yp - 3y^2 = 0 \quad \rightarrow \quad (p - 3y)(p + y) = 0$

$p - 3y = 0 \quad \rightarrow \quad \frac{dy}{dx} - 3y = 0 \quad \rightarrow \quad \frac{dy}{dx} = 3y$

$\frac{dy}{y} = 3dx \quad \rightarrow \quad lny = 3x + c_1 \quad \rightarrow \quad y = e^{3x+c_1}$

$y = Ae^{3x} \qquad where \quad A = e^{c_1}$

$p + y = 0 \rightarrow \quad \frac{dy}{dx} + y = 0 \quad \rightarrow \quad \frac{dy}{dx} = -y$

$\frac{dy}{y} = -dx \quad \rightarrow \quad lny = -x + c_2 \quad \rightarrow \quad y = e^{-x+c_2}$

$y = Be^{-x} \qquad where \quad B = e^{c_2}$

The general solution is $(y - Ae^{3x})(y - Be^{-x}\ ) = 0$

3- $p^2 - py = x^2 + xy \quad \rightarrow \quad p^2 - py - x^2 - xy = 0$

$p^2 - x^2 - py - xy = 0 \quad \rightarrow \quad (p + x)(p - x) - y(p + x) = 0$

$(p + x)(p - x - y) = 0$

$p + x = 0 \rightarrow \quad \frac{dy}{dx} + x = 0 \quad \rightarrow \quad \frac{dy}{dx} = -x$

$dy = -xdx \quad \rightarrow \quad y = -\frac{x^2}{2} + c_1$

$p - x - y = 0 \quad \rightarrow \quad \frac{dy}{dx} - x - y = 0$

$\frac{dy}{dx} - y = x \qquad P(x) = -1 \quad Q(x) = x$

$I(x) = e^{\int P(x)dx} \rightarrow I(x) = e^{\int -1\ dx} \rightarrow I(x) = e^{-x}$

$I(x).y = \int I(x).Q(x)\ dx \rightarrow e^{-x}.y = \int e^{-x}.x\ dx$

$x.y = -xe^{-x} - e^{-x} + c_2 \rightarrow y = -x - 1 + c_2 e^x$

The general solution is $\left(y + \frac{x^2}{2} - c_1\right)(y + x + 1 - c_2 e^x) = 0$

4- $xp^2 + (y - x)p - y = 0 \rightarrow xp^2 + yp - xp - y = 0$

$xp^2 - xp + yp - y = 0 \rightarrow xp(p - 1) + y(p - 1) = 0$

$(p - 1)(xp + y) = 0$

$p - 1 = 0 \rightarrow \frac{dy}{dx} - 1 = 0 \rightarrow \frac{dy}{dx} = 1$

$dy = dx \rightarrow y = x + c_1$

$xp + y = 0 \rightarrow x\frac{dy}{dx} + y = 0 \rightarrow x\frac{dy}{dx} = -y$

$\frac{dy}{y} = -\frac{dx}{x} \rightarrow lny = -lnx + c_2 \rightarrow y = \frac{1}{x}.e^{c_2}$

$y = \frac{A}{x}$ $\quad$ *where* $A = e^{c_2}$

The general solution is $(y - x + c_1)\left(y - \frac{A}{x}\right) = 0$

1.5.2. <u>Equation solvable for (y)</u>

$y = f(x, y, p)$ differential with respect to x to obtain $\frac{dy}{dx} = f(x, p, \frac{dp}{dx})$

Examples: find the general solution of the ODE

1- $y - 3x - 2lnp = 0 \rightarrow y = 3x + 2lnp$

$\frac{dy}{dx} = 3 + \frac{2}{p}\frac{dp}{dx} \rightarrow [p = 3 + \frac{2}{p}\frac{dp}{dx}](p)$

$p^2 = 3p + 2\frac{dp}{dx} \rightarrow p^2 - 3p = 2\frac{dp}{dx}$

$\frac{dp}{p^2-3p} = \frac{1}{2}dx \rightarrow \frac{1}{p^2-3p} = \frac{1}{p(p-3)}$

$\frac{1}{p(p-3)} = \frac{A}{p} + \frac{B}{p-3} \rightarrow \frac{1}{p(p-3)} = \frac{Ap-3A+Bp}{p(p-3)}$

$$1 = p - 3A + Bp$$

$$A + B = 0 \quad \rightarrow \quad A = -B$$

$$-3A = 1 \quad \rightarrow \quad A = \frac{-1}{3} \quad \rightarrow \quad B = \frac{1}{3}$$

$$\int \frac{\frac{-1}{3}}{p} + \int \frac{\frac{1}{3}}{p-3} = \frac{1}{2}dx \quad \rightarrow -\frac{1}{3}ln|p| + \frac{1}{3}\ln|p-3| = \frac{1}{2}x + c$$

$$\frac{1}{3}\ln|p-3| - \frac{1}{3}ln|p| += \frac{1}{2}x + c \quad \rightarrow \frac{1}{3}ln\left|\frac{p-3}{p}\right| = \frac{1}{2}x + c$$

$$ln\left|\frac{p-3}{p}\right| = \frac{3}{2}x + 3c \quad \rightarrow \quad \frac{p-3}{p} = e^{\frac{3}{2}x+3c}$$

$$1 - \frac{3}{p} = Ae^{\frac{3}{2}x} \qquad where \quad A = e^{3c}$$

$$\frac{3}{p} = 1 - Ae^{\frac{3}{2}x} \quad \rightarrow \quad p = \frac{3}{1-Ae^{\frac{3}{2}x}}$$

$$y - 3x - 2ln\left|\frac{3}{1-Ae^{\frac{3}{2}x}}\right| = 0$$

2- $y + px = p^2x^4 \quad \rightarrow \quad y = p^2x^4 - px$

$$\frac{dy}{dx} = p^2.4x^3 + 2x^4p\frac{dp}{dx} - p - x\frac{dp}{dx}$$

$$p = p^2.4x^3 + 2x^4p\frac{dp}{dx} - p - x\frac{dp}{dx}$$

$$2p - p^2.4x^3 - x\frac{dp}{dx} + 2x^4p\frac{dp}{dx} = 0$$

$$2p(1 - 2x^3p) + x\frac{dp}{dx}(1 - 2x^3p) = 0$$

$$(1 - 2x^3p)\left(2p + x\frac{dp}{dx}\right) = 0$$

$$1 - 2x^3p = 0 \quad \rightarrow \quad 2x^3p = 1 \quad \rightarrow \quad p = \frac{1}{2x^3}$$

$$y = (\frac{1}{2x^3})^2x^4 - \frac{1}{2x^3}x \quad \rightarrow \quad y = \frac{1}{4x^6}x^4 - \frac{1}{2x^3}x$$

$$y = \frac{1}{4x^2} - \frac{1}{2x^2} \quad \rightarrow \quad y = -\frac{1}{4x^2}$$

$$2p + x\frac{dp}{dx} = 0 \;\rightarrow\; x\frac{dp}{dx} = -2p$$

$$\int \frac{1}{p}dp = \int \frac{-2}{x}d\,x \;\rightarrow\; lnp = -2lnx + c$$

$$e^{\,lnp} = e^{-2lnx+c} \;\rightarrow\; p = \frac{A}{x^2} \;\; where \;\; A = e^c$$

$$y = (\frac{A}{x^2})^2\; x^4 - \frac{A}{x^2}\,x \;\rightarrow\; y = A^2 - \frac{A}{x}$$

3- $y = 3\sqrt{1+p^2} \;\rightarrow\; y = 3(1+p^2)^{\frac{1}{2}}$

$$\frac{dy}{dx} = \frac{3}{2}(1+p^2)^{\frac{-1}{2}}\frac{dp}{dx} \;\rightarrow\; p = \frac{3}{2}(1+p^2)^{\frac{-1}{2}}2p\,\frac{dp}{dx}$$

$$1 = 3(1+p^2)^{\frac{-1}{2}}\,\frac{dp}{dx} \;\rightarrow\; \frac{1}{(1+p^2)^{\frac{1}{2}}}dp = \frac{1}{3}dx$$

$$\frac{1}{\sqrt{1+p^2}}dp = \frac{1}{3}dx \;\rightarrow\; \sinh^{-1}p = \frac{1}{3}x + c$$

$$p = \sinh(\frac{1}{3}x + c) \;\rightarrow\; y = 3\sqrt{1 + \sinh^2(\frac{1}{3}x + c)}$$

1.5.3. <u>Equation solvable for (x)</u>

$y = f(x, y, p)$ differential with respect to y to obtain $\frac{dx}{dy} = f(y, p, \frac{dp}{dx})$

Examples: find the general solution of the ODE

1- $y = 2px + y^2p^3 \;\rightarrow\; 2px = y - y^2p^3$

$$x = \frac{y}{2p} - \frac{y^2p^2}{2} \;\rightarrow\; \frac{dx}{dy} = \frac{1}{2p} - \frac{y}{2p^2}\frac{dp}{dy} - \frac{p^2}{2}\,.2y - y^2p\frac{dp}{dy}$$

$$[\frac{1}{p} = \frac{1}{2p} - \frac{y}{2p^2}\frac{dp}{dy} - \frac{p^2}{2}\,.2y - y^2p\frac{dp}{dy}](p^2)$$

$$p = \frac{p}{2} - \frac{y}{2}\frac{dp}{dy} - yp^4 - y^2p^3\frac{dp}{dy}$$

$$p - \frac{p}{2} + \frac{y}{2}\frac{dp}{dy} + yp^4 + y^2p^3\frac{dp}{dy} = 0$$

$$\left(\frac{p}{2} + yp^4\right) + \left(\frac{y}{2}\frac{dp}{dy} + y^2p^3\frac{dp}{dy}\right) = 0$$

$$p\left(\frac{1}{2}+yp^3\right)+y\frac{dp}{dy}\left(\frac{1}{2}+yp^3\right)=0$$

$$\left(\frac{1}{2}+yp^3\right)\left(p+y\frac{dp}{dy}\right)=0$$

$$\frac{1}{2}+yp^3=0 \;\rightarrow\; yp^3=-\frac{1}{2} \;\rightarrow\; p^3=-\frac{1}{2y} \;\rightarrow\; p=\sqrt[3]{-\frac{1}{2y}}$$

$$y=2(\sqrt[3]{-\frac{1}{2y}})x+y^2(\sqrt[3]{-\frac{1}{2y}})^3 \rightarrow y=2(\sqrt[3]{-\frac{1}{2y}})x+y^2(-\frac{1}{2y})$$

$$y=2\left(\sqrt[3]{-\frac{1}{2y}}\right)x-\frac{y}{2} \;\rightarrow\; \frac{3y}{2}=2\left(\sqrt[3]{-\frac{1}{2y}}\right)x$$

$$\frac{27y^3}{8}=8\left(-\frac{1}{2y}\right)x^3 \;\rightarrow\; y^4=\left(\frac{32}{27}\right)x^3 \;\rightarrow\; y=\sqrt[4]{\left(\frac{32}{27}\right)x^3}$$

$$p+y\frac{dp}{dy}=0 \;\rightarrow\; y\frac{dp}{dy}=-p$$

$$\int\frac{1}{p}dp=\int\frac{-1}{y}dy \;\rightarrow\; lnp=-lny+c$$

$$e^{\,lnp}=e^{-lny+c} \;\rightarrow\; p=y^{-1}.e^{c}$$

$$p=\frac{A}{y} \qquad where \;\; A=e^{c}$$

$$y=2\left(\frac{A}{y}\right)x+y^2\left(\frac{A}{y}\right)^3 \;\rightarrow\; y=2A\frac{x}{y}+\frac{A^3}{y}$$

$$y^2=2Ax+A^3 \;\rightarrow\; y=\sqrt{2Ax+A^3}$$

2- $y=4px-16y^3p^2 \;\rightarrow\; [4px=y+16y^3p^2]\div 4p$

$$x=\frac{y}{4p}+4y^3p \;\rightarrow \frac{dx}{dy}=\frac{1}{4p}-\frac{y}{4p^2}\frac{dp}{dy}+12y^2p+4y^3\frac{dp}{dy}$$

$$\frac{1}{p}=\frac{1}{4p}-\frac{y}{4p^2}\frac{dp}{dy}+12y^2p+4y^3\frac{dp}{dy}$$

$$[\frac{1}{p}-\frac{1}{4p}+\frac{y}{4p^2}\frac{dp}{dy}-12y^2p-4y^3\frac{dp}{dy}=0](p^2)$$

$$[p-\frac{p}{4}+\frac{y}{4}\frac{dp}{dy}-12y^2p^3-4y^3p^2\frac{dp}{dy}](4)$$

$$4p-p+y\frac{dp}{dy}-48y^2p^3-16y^3p^2\frac{dp}{dy}$$

$$(3p - 48y^2p^3)\left(y\frac{dp}{dy} - 16y^3p^2\frac{dp}{dy}\right) = 0$$

$$3p(1 - 16y^2p^2) + y\frac{dp}{dy}(1 - 16y^2p^2) = 0$$

$$(1 - 16y^2p^2)\left(3p + y\frac{dp}{dy}\right) = 0$$

$$1 - 16y^2p^2 = 0 \;\;\rightarrow\;\; 16y^2p^2 = 1$$

$$p^2 = \frac{1}{16y^2} \;\;\rightarrow\;\; p = \frac{1}{4y}$$

$$y = 4\left(\frac{1}{4y}\right)x - 16y^3\left(\frac{1}{4y}\right)^2 \;\;\rightarrow\;\; y = \frac{x}{y} - 4y$$

$$5y = \frac{x}{y} \;\;\rightarrow\;\; 5y^2 = x \;\;\rightarrow\;\; y = \sqrt{\frac{x}{5}}$$

$$3p + y\frac{dp}{dy} = 0 \;\;\rightarrow\;\; y\frac{dp}{dy} = -3p$$

$$\int \frac{1}{p}dp = \int \frac{-3}{y}dy \;\;\rightarrow\;\; lnp = -3lny + c$$

$$e^{\,lnp} = e^{-3lny+c} \;\;\rightarrow\;\; p = y^{-3}.e^{c}$$

$$p = \frac{A}{y^3} \qquad\qquad where \;\; A = e^{c}$$

$$y = 4\left(\frac{A}{y^3}\right)x - 16y^3\left(\frac{A}{y^3}\right)^2 \;\;\rightarrow\;\; y = 4Ax\left(\frac{1}{y^3}\right) - 16A\left(\frac{1}{y^3}\right)$$

$$y^4 = 4Ax - 16A \;\;\rightarrow\;\; y = \sqrt[4]{4Ax - 16A}$$

H.W. Find the general solution of the differential equation

1-- $4x^2p^2 + 2y^2 = 6xyp$ 2-- $x = p(1 + xy) - yp^2$

3-- $2xy(3x + 1)p = 3x^3 + 4y^2p^2$

4-- $6xy + p(2y^2 - 3x^2) - xyp^2 = 0$

5-- $p\sin x + \cos x$ 6-- $x^2p^4 = y - 2xp$

7-- $p^2 = y - x(p + 1)$ 8-- $2xyp = 4y^2 + p^3$

9-- $y^2p^2 = y - 3px$

10-- $x = y - 2\tan^{-1} p$

11-- $p = \tan\left(x - \frac{p}{1+p^2}\right)$

12--$p^2 = y^2 lny - xyp$