<u>Linear differential equation of order (n)</u>

A Linear differential equation of order (n) has the form

$$a_n \frac{d^n y}{dx^n} + a_{n-1} \frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_0 y = f(x) \ldots\ldots\ldots (1) \quad \text{Where } a_n \neq 0$$

If $f(x) = 0$ then (1) has the form

$$a_n \frac{d^n y}{dx^n} + a_{n-1} \frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_0 y = 0 \ldots\ldots\ldots..\ldots (2)$$

Is called homogenous to indicate that all the terms are of the same (first) degree in (y) and its derivatives

Linear Differential Operator: We define a linear differential Operator

Say (L) as follows:

$$L = a_n \frac{d^n}{dx^n} + a_{n-1} \frac{d^{n-1}}{dx^{n-1}} + \cdots + a_0$$

Now

$$L(y) = a_n \frac{d^n y}{dx^n} + a_{n-1} \frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_0 y$$

Or

$$L(y) = a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_0 y$$

Hence, equation (1) can be written in the form.

$$L(y) = f(x) \ldots\ldots\ldots\ldots (3)$$

Theorem1(linear property)**: -** If $y_1(x)$ and $y_2(x)$ are two functions, then $L[c_1 y_1(x) + c_2 y_2(x)] = c_1 L(y_1(x)) + c_2 L(y_2(x))$ where c_1 and c_2 are constants

Proof: - in this case $L(y)$ can be written in the following

$$L(y) = y''(x) + a y'(x) + b y(x)$$

Then $L[c_1y_1(x) + c_2y_2(x)] = [c_1y_1(x) + c_2y_2(x)]'' + a[c_1y_1(x) + c_2y_2(x)]' + b[c_1y_1(x) + c_2y_2(x)]$

$= c_1y_1'' + c_2y_2'' + ac_1y_1' + ac_2y_2' + bc_1y_1 + bc_2y_2$

$= (c_1y_1'' + ac_1y_1' + bc_1y_1) + (c_2y_2'' + ac_2y_2' + bc_2y_2)$

$= c_1(y_1'' + ay_1' + bc_1y_1) + c_2(y_2'' + ay_2' + by_2)$

$= c_1L(y_1) + c_2L(y_2)$

Theorem2: - If y_1 and y_2 are two solutions of the homogeneous linear DE $L(y) = 0$ then $y = c_1y_1 + c_2y_2$ is also a solution of $L(y) = 0$

Proof: - since y_1 and y_2 are two solutions of $L(y) = 0$ then $L(y_1) = 0$ and $L(y_2) = 0$

Now $L[c_1y_1 + c_2y_2] = c_1L(y_1) + c_2L(y_2) = c_1.0 + c_2.0 = 0$

then $y = c_1y_1 + c_2y_2$ is also a solution.

Definition: - let $y_1, y_2,\ldots\ldots, y_n$ are all functions that have $(n-1)$ derivatives on an interval $a < x < b$, the Wronskian of the functions $y_1, y_2,\ldots\ldots, y_n$ written (w) is a determined defined by

$$w = \begin{vmatrix} y_1 & y_2 & \cdots & y_n \\ y_1' & y_2' & \cdots & y_n' \\ y_1'' & y_2'' & \cdots & y_n'' \\ \vdots & \vdots & \vdots & \vdots \\ y_1^{(n-1)} & y_2^{(n-1)} & \cdots & y_n^{(n-1)} \end{vmatrix}$$

Theorem3: - suppose that $y_1, y_2, \ldots\ldots, y_n$ is a set of (n) solutions of the n-th order linear homogeneous DE $L(y) = 0$, then the set of functions that $y_1(x), y_2(x), \ldots\ldots, y_n(x)$ is linearly independent of

$a < x < b$ iff the Wronskian of the set is not zero.

Examples: the Wronskian shows that

1- $\{1, x, x^2\}$ are linearly independent

$$w = \begin{vmatrix} 1 & x & x^2 \\ 0 & 1 & 2x \\ 0 & 0 & 2 \end{vmatrix} = (1)\begin{vmatrix} 1 & 2x \\ 0 & 2 \end{vmatrix} - (x)\begin{vmatrix} 0 & 2x \\ 0 & 2 \end{vmatrix} + (x^2)\begin{vmatrix} 0 & 1 \\ 0 & 0 \end{vmatrix} = 2 \neq 0$$

2- $y_1 = e^{-x}, y_2 = e^{2x}$

$$w = \begin{vmatrix} e^{-x} & e^{2x} \\ -e^{-x} & 2e^{2x} \end{vmatrix} = 2e^x + e^x = 3e^x \neq 0$$

3- $c_1 y_1 = -c_2 y_2 \quad \rightarrow \quad y_2 = \frac{c_1}{c_2} y_1$

Let $A = \frac{c_1}{c_2} \quad \rightarrow \quad y_2 = A y_1$

$$w = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} y_1 & Ay_1 \\ y_1' & Ay_1' \end{vmatrix} = y_1 y_1' - A y_1 y_1' = 0$$

Theorem4: - if $L(y)$ is a linear n-th order homogeneous DE, then it has (n) linearly independent solution say $y_1(x), y_2(x), \ldots\ldots, y_n(x)$ Then the general solution of $L(y)$ is a linear combination of those solutions that is: - $y = c_1 y_1(x) + c_2 y_2(x) + \cdots + c_n y_n(x)$ where $c_1, c_2, \ldots, c_n$ are arbitrary constants

Theorem5: the general solution of the n-th order linear non-homogeneous DE $L(y) = f(x)$ is consisted

1- the general solution of the homogeneous DE $L(y) = 0$ id denoted by y_c
2- a special solution of non-homogeneous DE $L(y) = f(x)$ written y_p and is called the particular solution of DE, then $y = y_c + y_p$

homogeneous linear differential equation with constant coefficient

A homogeneous linear differential equation with constant coefficients has the form

$a_n \frac{d^n y}{dx^n} + a_{n-1} \frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_0 y = 0 \dots \dots \dots (1)$ Where $a_n \neq 0$

And $a_{n-1}, a_{n-2}, \dots \dots, a_0$ are all constants.

Let $D = \frac{d}{dx}$, $Dy = \frac{dy}{dx}$, $D^2 y = \frac{d^2 y}{dx^2}$, … … , $D^n y = \frac{d^n y}{dx^n}$

Then equation (1) becomes

$a_n D^n y + a_{n-1} D^{n-1} y + \cdots + a_0 y = 0 \dots \dots \dots (2)$

Or

$(a_n D^n + a_{n-1} D^{n-1} + \cdots + a_0) y = 0 \dots \dots \dots (3)$

Define the operator $L(y) = a_n D^n + a_{n-1} D^{n-1} + \cdots + a_0$

Hence, equation (3) becomes $L(D)y = 0 \dots \dots \dots (4)$

To find the general solution, let $y = e^{mx}$ be any solution (m is constant) now $L(D). e^{mx} = 0 \rightarrow L(m). e^{mx} = 0 \rightarrow L(m) = 0$, which is the characteristic equation of (1)

Case 1: if $m_1 \neq m_2 \neq, \dots \dots \dots \dots, m_n$ (real distant roots) the general solution $y = c_1 e^{m_1 x} + c_2 e^{m_2 x} + \cdots + c_n e^{m_n x}$ where $c_1, c_2, \dots, c_n$ are constant $y_1 = e^{m_1 x}, y_2 = e^{m_2 x}, \dots \dots, y_n = e^{m_n x}$

Case 2: if $m_1 = m_2 =, \dots \dots \dots \dots, m_n$ (real equal roots) the general solution $y = c_1 e^{m_1 x} + c_2 x e^{m_2 x} + \cdots + c_n x^{n-1} e^{m_n x}$ where $c_1, c_2, \dots, c_n$ are constant $y_1 = e^{m_1 x}, y_2 = x e^{m_2 x}, \dots \dots, y_n = x^{n-1} e^{m_n x}$

Case 3: for any pair of imaginary roots, say $m_1 = a + bi\ , m_2 = a - bi$

The general solution is $y = e^{ax}(c_1 \cos bx + c_2 \sin bx)$ and $y_1 = e^{ax} c_1 \cos bx \quad , \quad y_2 = e^{ax} c_2 \sin bx$

Examples: find the general solution of the ODE

1- $2\frac{d^2y}{dx^2} + 5\frac{dy}{dx} - 3y = 0$

$2D^2y + 5Dy - 3y = 0$

$(2D^2 + 5D - 3)y = 0$

The characteristic equation is

$2m^2 + 5m - 3 = 0$

$(m + 3)(2m - 1) = 0$

$m + 3 = 0 \quad \rightarrow \quad m = -3$

$2m - 1 = 0 \quad \rightarrow \quad 2m = 1 \quad \rightarrow \quad m = \frac{1}{2}$

the general solution $y = c_1 e^{m_1 x} + c_2 e^{m_2 x}$

$y = c_1 e^{3x} + c_2 e^{\frac{1}{2}x}$

$y_1 = e^{3x} \quad , \quad y_2 = e^{\frac{1}{2}x}$

2- $y'' + 4y' + y = 0$

$D^2y + 4Dy + y = 0$

$(D^2 + 4D + 1)y = 0$

The characteristic equation is

$m^2 + 4m + 1 = 0$

$m_{1,2} = \frac{-b \pm \sqrt{b^2-4ac}}{2a} \quad \rightarrow \quad m_{1,2} = \frac{-4 \pm \sqrt{16-4}}{2}$

$m_{1,2} = \frac{-4 \pm \sqrt{12}}{2} \quad \rightarrow \quad m_{1,2} = \frac{-4 \pm 2\sqrt{3}}{2}$

$m_{1,2} = -2 \pm 2\sqrt{3}$

$m_1 = -2 + 2\sqrt{3}$, $m_2 = -2 - 2\sqrt{3}$

the general solution $y = c_1 e^{m_1 x} + c_2 e^{m_2 x}$

$y = c_1 e^{(-2+2\sqrt{3})x} + c_2 e^{(-2-2\sqrt{3})x}$

$y_1 = e^{(-2+2\sqrt{3})x}$, $y_2 = e^{(-2-2\sqrt{3})x}$

3- $(D^3 + 4D^2 + 4D)y = 0$

The characteristic equation is

$m^3 + 4m^2 + 4m = 0$

$m(m^2 + 4m + 4) = 0$

$m(m + 2)(m + 2) = 0$

$m = 0$, $m = -2$, $m = -2$

the general solution $y = c_1 e^{m_1 x} + c_2 e^{m_2 x} + c_3 x e^{m_3 x}$

$y = c_1 e^{0x} + c_2 e^{-2x} + c_3 x e^{-2x}$

$y_1 = 1,\quad y_2 = e^{-2x},\quad y_3 = xe^{-2x}$

4- $(D^3 + D^2 - 3D + 1)y = 0$

The characteristic equation is

$m^3 + m^2 - 3m + 1 = 0$

$m_1 = 1 \quad \rightarrow \quad m - 1 = 0$

$(m - 1)(m^2 + 2m - 1) = 0$

$m_{2,3} = \frac{-b\pm\sqrt{b^2-4ac}}{2a} \quad \rightarrow \quad m_{2,3} = \frac{-2\pm\sqrt{4+4}}{2}$

$m_{1,2} = \frac{-2\pm\sqrt{8}}{2} \quad \rightarrow \quad m_{1,2} = \frac{-2\pm2\sqrt{2}}{2}$

$m_{1,2} = -1 \pm \sqrt{2}$

$m_1 = -1 + \sqrt{2} \;\;,\;\; m_2 = -1 - \sqrt{2}$

the general solution $y = c_1 e^{m_1 x} + c_2 e^{m_2 x} + c_3 e^{m_3 x}$

$y = c_1 e^{x} + c_2 e^{(-1+\sqrt{2})x} + c_3 e^{(-1-\sqrt{2})x}$

$y_1 = e^{x} \;\;,\;\; y_2 = e^{(-1+\sqrt{2})x}, \;\; y_3 = e^{(-1-\sqrt{2})x}$

5- $(D^4 - 8D^2 + 16)y = 0$

The characteristic equation is

$m^4 - 8m^2 + 16 = 0$

$(m^2 - 4)(m^2 - 4) = 0$

$(m - 2)(m + 2)(m - 2)(m + 2) = 0$

$m_{1,3} = -2, \; m_{2,4} = 2$

the general solution $y = c_1 e^{m_1 x} + c_3 x e^{m_3 x} + c_2 e^{m_2 x} + c_4 x e^{m_4 x}$

$y = c_1 e^{-2x} + c_3 x e^{-2x} + c_2 e^{2x} + c_4 x e^{2x}$

$y_1 = e^{-2x}, \;\; y_3 = x e^{-2x} y_2 = e^{2x}, \;\; y_4 = x e^{2x}$

6- $y'' + 4y' + 5y = 0$

$D^2 y + 4Dy + 5y = 0$

$(D^2 + 4D + 5)y = 0$

The characteristic equation is

$m^2 + 4m + 5 = 0$

$m_{1,2} = \frac{-b\pm\sqrt{b^2-4ac}}{2a} \quad \rightarrow \quad m_{1,2} = \frac{-4\pm\sqrt{16-20}}{2}$

$m_{1,2} = \frac{-4\pm\sqrt{-4}}{2} \quad \rightarrow \quad m_{1,2} = \frac{-4\pm2i}{2}$

$m_{1,2} = -2 \pm i \qquad m_1 = -2 + i \;\;,\;\; m_2 = -2 - i$

The general solution is $y = e^{ax}(c_1 \cos bx + c_2 \sin bx)$

$y = e^{-2x}(c_1 \cos x + c_2 \sin x)$

$y_1 = e^{-2x} c_1 \cos x \quad , \quad y_2 = e^{-2x} c_2 \sin x$

H.W. Find the general solution of the differential equation

1-- $y'' + 2y' - 15y = 0$

2-- $y'' - 2y' - 5y = 0$

3-- $3y'' + 2y' + y = 0$

4-- $y'' - 4y' + 7y = 0$

5-- $(D^3 - 6D^2 + 12D - 8)y = 0$

6-- $(D^3 + D^2 + D + 1)y = 0$

7-- $(D^4 + 2D^3 + D^2)y = 0$

8-- $(D^3 + 2D^2 + D + 2)y = 0$

9-- $(4D^3 + 6D^2 - 6D - 4)y = 0$

10-- $(D^3 - 6D^2 + 11D - 6)y = 0$

11-- $(6D^3 + 11D^2 - 12D - 5)y = 0$