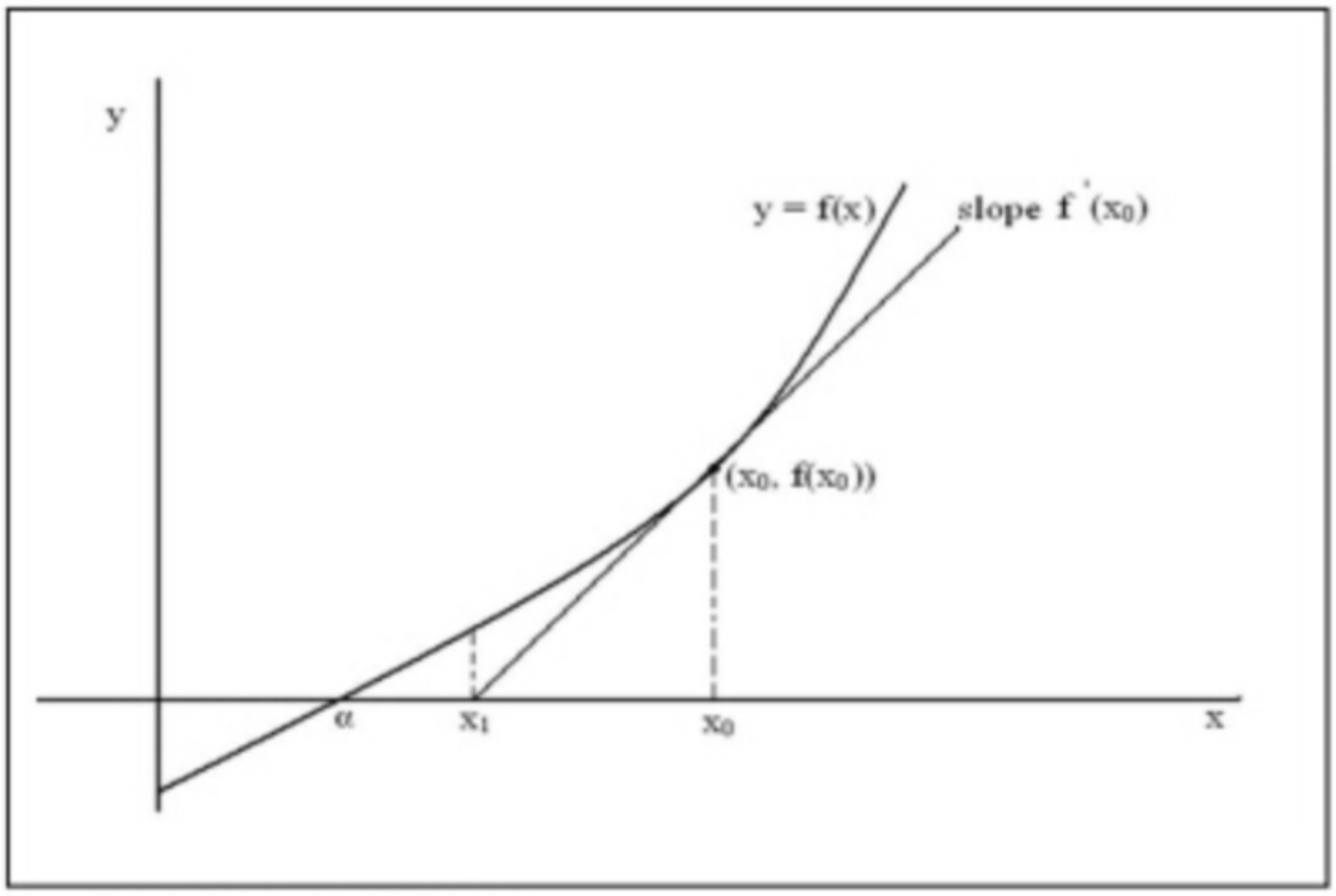
C. Newton – Raphson Method

Newton – Raphson (or Newton) method can be used to approximate the roots of any linear or non – linear equation of any degree.



We can calculate the slope using the fact that the tangent line contains the two points (x_0, $f(x_0)$) and (x_1, 0) this leads to the slope being equal to:-

$$\frac{0 - f(x_0)}{x_1 - x_0} = slop = f'(x_0)$$

Then

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

And in general,

$$\boxed{x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}} \quad n = 0,1,2,3, \ldots$$

Condition of stop

Using some mathematical formulas to stop doing calculations for this method.

1- $|x_{i+1} - x_i| \leq \epsilon$

Convergence of Newton Raphson method.

$$\left|\frac{f(x_0)f''(x_0)}{(f'(x_0))^2}\right| < 1$$

Example: By N. R method find the solution of the following equations.

$$f(x) = 2x - 1 - 2sinx, \qquad where\ x_0 = 2\ and\ \epsilon = 0$$

Solution:

$$f(x) = 2x - 1 - 2\sin(x) \Rightarrow f(x_0) = f(2) = 2(2) - 1 - 2\sin(2)$$
$$= 1.181$$

$f'(x) = 2 - 2\cos(x) \Rightarrow f'(x) = f'(2) = 2 - \cos(2) =$

2.832

$$f''(x) = 2\,s\,in(x) \Rightarrow f'(x) = f'(2) = 2\sin(2) = 1.819$$

$$\left|\frac{f(x_0)f''(x_0)}{(f'(x_0))^2}\right| = \left|\frac{(1.181)(1.819)}{(2.832)^2}\right| = 0.268 < 1$$

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 2 - \frac{1.181}{2.832} = 1.583$$

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = 1.583 - \frac{0.166}{2.024} = 1.501$$

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)} = 1.501 - \frac{0.007}{1.861} = 1.497$$

$$x_4 = x_3 - \frac{f(x_3)}{f'(x_3)} = 1.497 - \frac{0.001}{1.853} = 1.498$$

$$x_5 = x_4 - \frac{f(x_4)}{f'(x_4)} = 1.498 - \frac{0.001}{1.855} = 1.498$$

Since $|x_5 - x_4| = 0$ then $x_5 = 1.498$ is a root.

Example

Solve the following equation using Newton method

$$f(x) = x^6 - x - 1, \quad where \ \epsilon = 0$$

x	0	1	2

$f(x)$	-	-	+

Solution

$$x_0 = \frac{1+2}{2} = 1.5$$

$$f(x) = x^6 - x - 1 \Rightarrow f(1.5) = (1.5)^6 - 1.5 - 1 = 8.890625$$

$$f'(x) = 6x^5 - 1 \Rightarrow f'(1.5) = 6(1.5)^5 - 1 = 44.5625$$

$$f''(x) = 30x^4 \Rightarrow f''(1.5) = 30(1.5)^4 = 151.875$$

$$\left|\frac{f(x_0)f''(x_0)}{(f'(x_0))^2}\right| = \left|\frac{(8.890625)(151.875)}{(44.5625)^2} = \frac{1350.320625}{1985.81640625}\right|$$

$$= 0.679982 < 1$$

The iteration is given by:

$$x_{n+1} = x_n - \frac{f(x)}{f'(x)}$$

n	x_n	$f(x_n)$	$\alpha - x_n$
0	1.5	8.8906	-0.1996
1	1.3004	2.5353	-0.119
2	1.1814	0.5374	-0.042
3	1.1394	0.0487	$-4.6*10^{-3}$
4	1.1348	$7.8058*10^{-4}$	$-1*10^{-4}$
5	1.1347	$-2.4831*10^{-4}$	0
6	1.1347	$-2.4831*10^{-4}$	0

$$x_{n+1} = x_n - \left(\frac{x_n^6 - x_n - 1}{6x_n^5 - 1} \right)$$

The true root α = 1.1347 and x is equal α.

The advantage of the Newton – Raphson method:

1. Guaranteed to converge to a root if x_0 is close enough to it and f is sufficiently smooth
2. Quadratic convergence near simple root
3. Works for complex functions

The disadvantage of the Newton – Raphson method:

1. Iterates may diverge
2. Requires derivative
3. No easy error bound

<u>H.W:</u>

1. $x^3 - x^2 - x - 1 = 0$, $x_0 = 1.5$
2. $\ln(x-1) + \cos(x-1) = 0$, $x_0 = 1.1$