## 2. Backward Differences (الخلفية) الفروقات التراجعية

يرمز لعامل (مؤثر) الفروق التراجعية بالرمز $\nabla$ operator ويعرف الفرق التراجعي للدالة $f(x)$ بالمعادلة الاتية

$$\nabla f(x) = f(x) - f(x+h)$$

ويمكن تعريف الفروقات التراجعية بالاتي:

$$\nabla y_i = y_i - y_{i-1} \qquad i = n, \dots, 1$$

$$\nabla^2 y_i = \nabla(\nabla y_i) = \nabla(y_i - y_{i-1}) = \nabla y_i - \nabla y_{i-1} \qquad i = n, \dots 2$$
$$= y_i - y_{i-1} - y_{i-1} + y_{i-2}$$
$$= y_i - 2y_{i-1} + y_{i-2}$$

$$\nabla^3 y_i = \nabla(\nabla^2 y_i) = \nabla(y_i - 2y_{i-1} + y_{i-2}) = \nabla y_i - 2\nabla y_{i-1} + \nabla y_{i-2}$$
$$= y_i - y_{i-1} - 2(y_{i-1} - y_{i-2}) + y_{i-2} - y_{i-3}$$
$$= y_i - y_{i-1} - 2y_{i-1} + 2y_{i-2} + y_{i-2} - y_{i-3}$$
$$= y_i - 3y_{i-1} - 3y_{i-1} - y_{i-3} \qquad i = n, \dots, 3$$

Or

$$\nabla^3 y_i = \nabla^2(\nabla y_i) = \nabla^2 y_i - \nabla^2 y_{i-1}$$
$$\nabla^4 y_i = \nabla^3(\nabla y_i) = \nabla^3 y_i - \nabla^3 y_{i-1}$$

يمكن الحصول على كل الفروقات التراجعية من جدول بسيط يكون كما اناه

| $x_i$ | $y_i$ | $\nabla y_i$ | $\nabla^2 y_i$ | $\nabla^3 y_i$ | $\nabla^4 y_i$ |
|---|---|---|---|---|---|
| $x_0$ | $y_0$ | | | | |
| | | $\nabla y_1$ | | | |
| $x_1$ | $y_1$ | | $\nabla^2 y_2$ | | |
| | | $\nabla y_2$ | | $\nabla^3 y_3$ | |
| $x_2$ | $y_2$ | | $\nabla^2 y_3$ | | $\nabla^4 y_4$ |
| | | $\nabla y_3$ | | $\nabla^3 y_4$ | |
| $x_3$ | $y_3$ | | $\nabla^2 y_4$ | | |
| | | $\nabla y_4$ | | | |
| $x_4$ | $y_4$ | | | | |

**Example:** write differences table of the function $f(x) = x^3$, use $(x = 0,1,2,3,4)$

| $x_i$ | $y_i$ | $\nabla y_i$ | $\nabla^2 y_i$ | $\nabla^3 y_i$ | $\nabla^4 y_i$ |
|---|---|---|---|---|---|
| 0 | 0 | | | | |
| | | 1 | | | |
| 1 | 1 | | 6 | | |
| | | 7 | | 6 | |
| 2 | 8 | | 12 | | 0 |
| | | 19 | | 6 | |
| 3 | 27 | | 18 | | |
| | | 37 | | | |
| 4 | 64 | | | | |

## 3. Central Differences (الوسطى) الفروقات المركزية

**يرمز لمؤثر الفروقات المركزية بالرمز** $\delta$ وتعرف الفروقات المركزية بالاتي

**الفرق الاول**

$$\delta_{i+\frac{1}{2}} = y_{i+1} - y_i \qquad i = 0, \dots, n-1$$

$$\text{if } i = 0 \implies \delta_{\frac{1}{2}} = y_1 - y_0$$

$$i = 1 \implies \delta_{\frac{1}{2}} = y_2 - y_1$$

$$i = 2 \implies \delta_{\frac{5}{2}} = y_3 - y_2$$

**الفرق الثاني**

$$\delta^2 y_i = \delta(\delta y_i) = \delta\left(y_{i+\frac{1}{2}} - y_{i-\frac{1}{2}}\right) = \delta y_{i+\frac{1}{2}} - \delta y_{i-\frac{1}{2}}$$

$$= (y_{i+1} - y_i - (y_i - y_{i-1}))$$

$$= y_{i+1} - 2y_i + y_{i-1}$$

$$\text{if } i = 0 \implies \delta^2 y_0 = \delta y_{\frac{1}{2}} - \delta y_{-\frac{1}{2}} = y_1 - y_0 - (y_0 - y_{-1})$$

$$= y_1 - 2y_0 + y_{-1}$$

$$i = 1 \implies \delta^2 y_1 = \delta y_{\frac{3}{2}} - \delta y_{\frac{1}{2}} = y_2 - y_1 - (y_1 - y_0)$$

$$= y_2 - 2y_1 + y_0$$

$$i = 2 \implies \delta^2 y_2 = \delta y_{\frac{5}{2}} - \delta y_{\frac{3}{2}} = y_0 - y_{-1} - (y_{-1} - y_{-2})$$

$$= y_0 - 2y_{-1} + y_{-2}$$

الفرق الثالث

$$\delta^3 y_{i+\frac{1}{2}} = \delta^2 y_{i+1} - \delta^2 y_i$$

$$\text{If } i = 0 \implies \delta^3 y_{\frac{1}{2}} = \delta^2 y_1 - \delta^2 y_0$$

$$i = 1 \implies \delta^3 y_{\frac{3}{2}} = \delta^2 y_2 - \delta^2 y_1$$

$$i = 2 \implies \delta^3 y_{\frac{5}{2}} = \delta^2 y_3 - \delta^2 y_2$$

$$\delta^4 y_{\frac{1}{2}} = \delta^3 y_{i+\frac{1}{2}} - \delta^3 y_{i-\frac{1}{2}}$$

$$i = 0 \implies \delta^4 y_0 = \delta^3 y_{\frac{1}{2}} - \delta^3 y_{-\frac{1}{2}}$$

$$i = 1 \implies \delta^4 y_1 = \delta^3 y_{\frac{3}{2}} - \delta^3 y_{\frac{1}{2}}$$

$$i = 2 \implies \delta^4 y_2 = \delta^3 y_{\frac{5}{2}} - \delta^3 y_{\frac{3}{2}}$$

ويمكن تعميم الفروقات المركزية من الرتبة k بالاتي

$$\delta^{2k+1} y_{i+\frac{1}{2}} = \delta^{2k} y_{i+1} - \delta^{2k} y_i \quad i = 0, \dots, \quad k = 0,1,2 \qquad \text{فردي}$$

$$\delta^{2k} y_i = \delta^{2k-1} y_{i+\frac{1}{2}} - \delta^{2k-1} y_{i-\frac{1}{2}} \quad i = 0, \dots, \quad k = 1,2,3 \qquad \text{زوجي}$$

ويمكن الحصول على جميع الفروقات من جدول بسيط في ادناه

| $x$ | $y$ | $\delta y_{i+\frac{1}{2}}$ | $\delta^2 y_i$ | $\delta^3 y_{i+\frac{1}{2}}$ | $\delta^4 y_i$ |
|---|---|---|---|---|---|
| $x_{-1}$ | $y_{-1}$ | | $\delta^2 y_{-1}$ | | $\delta^4 y_{-1}$ |
| | | $\delta y_{-\frac{1}{2}}$ | | $\delta^3 y_{-\frac{1}{2}}$ | |
| $x_0$ | $y_1$ | | $\delta^2 y_0$ | | $\delta^4 y_0$ |
| | | $\delta y_{\frac{1}{2}}$ | | $\delta^3 y_{\frac{1}{2}}$ | |
| $x_1$ | $y_1$ | | $\delta^2 y_1$ | | $\delta^4 y_1$ |
| | | $\delta y_{\frac{3}{2}}$ | | $\delta^3 y_{\frac{3}{2}}$ | |
| $x_2$ | $y_2$ | | $\delta^2 y_2$ | | $\delta^4 y_2$ |
| | | $\delta y_{\frac{5}{2}}$ | | $\delta^3 y_{\frac{5}{2}}$ | |
| $x_3$ | $y_3$ | | $\delta^2 y_3$ | | $\delta^4 y_3$ |
| | | $\delta y_{\frac{7}{2}}$ | | $\delta^3 y_{\frac{7}{2}}$ | |
| $x_4$ | $y_4$ | | $\delta^2 y_4$ | | $\delta^4 y_4$ |

central

$$\delta_{\frac{1}{2}} = y_1 - y_0$$

$$\delta_{11\frac{1}{2}} = y_6 - y_5$$


$$\Delta y_0 = y_1 - y_0$$

$$\nabla y_6 = y_6 - y_5$$

Forward

backward

| $x$ | $y$ | | | |
|---|---|---|---|---|
| $x_{-2}$ | $y_{-2}$ | $\Delta y_{-2}, \nabla y_{-1}, \delta y_{-\frac{3}{2}}$ | $\Delta^2 y_{-3} \nabla^2 y_{-1}, \delta^2 y_{-2}$ | $\Delta^3 y_{-3}, \nabla^3 y_0, \delta^3 y_{-\frac{3}{2}}$ |
| $x_{-1}$ | $y_{-1}$ | $\Delta y_{-1}, \nabla y_0, \delta y_{-\frac{1}{2}}$ | $\Delta^2 y_{-2}, \nabla^2 y_0, \delta^2 y_{-1}$ | $\Delta^3 y_{-2} \nabla^3 y_1, \delta^3 y_{-\frac{1}{2}}$ |
| $x_0$ | $y_0$ | $\Delta y_0, \nabla y_1, \delta y_{\frac{1}{2}}$ | $\Delta^2 y_{-1}, \nabla^2 y_1, \delta^2 y_0$ | $\Delta^3 y_{-1} \nabla^3 y_2, \delta^3 y_{\frac{1}{2}}$ |
| $x_1$ | $y_1$ | $\Delta y_1, \nabla y_2, \delta y_{\frac{3}{2}}$ | $\Delta^2 y_0 \nabla^2 y_2, \delta^2 y_1$ | $\Delta^3 y_0 \nabla^3 y_3, \delta^3 y_{\frac{3}{2}}$ |
| $x_2$ | $y_2$ | | $\Delta^2 y_1 \nabla^2 y_3, \delta^2 y_2$ | |

من ملاحظة الجدول اعلاه يمكن ايجاد العلاقة بين المؤثرات الثلاثة

الفرق الاول $\Delta y_i = \nabla y_{i+1} = \delta y_{i+\frac{1}{2}}$

الفرق الثاني $\Delta^2 y_i = \nabla^2 y_{i+2} = \delta^2 y_{i+1}$

الفرق الثالث $\Delta^3 y_i = \nabla^3 y_{i+3} = \delta^3 y_{i+\frac{3}{2}}$

والصيغة العامة هي k=1,2,3,... $\Delta^k y_i = \nabla^k y_{i+k} = \delta^k y_{i+\frac{k}{2}}$

**Example:** write differences table of the function $f(x) = e^x + 2$, use $(x = 0,2,4,6,8)$ and find

$$\Delta y_2, \Delta^4 y_0, \nabla y_1, \nabla^2 y_2, \delta^2 y_1, \delta^4 y_2, \Delta^3 y_0 \nabla^3 y_4, \delta^3 y_{\frac{3}{2}}$$

| $x_i$ | $y_i$ | | | | |
|---|---|---|---|---|---|
| 0 | 3 | | | | |
| | | 6.389 | | | |
| 2 | 9.389 | | 640.820 | | |
| | | 47.209 | | 260.802 | |
| 4 | 56.598 | | 301.622 | | 1666.274 |
| | | 348.831 | | 1927.076 | |
| 6 | 405.429 | | 2228.698 | | |
| | | 2577.529 | | | |
| 8 | 2982.958 | | | | |

$$\Delta y_2 = 348.831$$

$$\Delta^4 y_0 = \delta^4 y_2 = 1666.274,$$

$$\nabla y_1 = 6.389,$$

$$\nabla^2 y_2 = \delta^2 y_1 = 40.820,$$

$$\nabla^3 y_4 = 1927.076$$

$$\Delta^3 y_0 = \delta^3 y_{\frac{3}{2}} = 260.802$$