NUMERICAL SOLUTIONS FOR ORDINARY DIFFERENTIAL EQUATIONS

Many differential equations are difficult to solve in analytical methods. In such equations, it is difficult to find the $y(x)$ function to achieve the differential equation. Numerical methods can be depended to find an approximate solution to such equations.

The object of the numerical methods is to obtain approximations to the **Initial Value Problem** (IVP):

$$\frac{dy}{dx} = f(x, y) \qquad a \leq x \leq b, \quad y(a) = \alpha$$

$$\text{where} \quad x_i = a + ih$$

EULER'S METHOD

We know that the derivative of the function $y = f(x)$ is given by: $y' = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$ and to solve the differential equation $y' = f(x, y)$. We assume that $\Delta y/\Delta x$ is an approximation of y' i.e.:

$$y' = \frac{\Delta y}{\Delta x}$$

$$\Delta y_k = y_{k+1} - y_k\,, \qquad \Delta x = x_{k+1} - x_k = h$$

$$y_{k+1} = y_k + \Delta y_k$$

From the relation $\frac{\Delta y}{\Delta x}$, we can write

$$y_k' = \frac{\Delta y_k}{\Delta x}$$

$$\Delta y_k = \Delta x y_k' = h y_k' = hf(x, y)$$

And hence

$$\boldsymbol{y_{k+1} = y_k + hf(x, y); \qquad \mathbf{k} = 0, 1, 2, \ldots}$$

This is Euler's formula.

Example 1: Apply Euler's method to solve the following differential equation (DE):

$$\frac{dy}{dx} = x + y \ , \quad y(0) = 1, \quad 0 \le x \le 0.10, \quad h = 0.02$$

Solution:

$x_0 = 0 \ , \quad y_0 = y(0) = 1$

$$y_1 = y_0 + hf(x_0, y_0) = 1 + 0.02 \times 1 = 1.02$$

$x_1 = 0.02$

$$y_2 = y_1 + hf(x_1, y_1) = 1.02 + 0.02 \times 1.04 = 1.0408$$

$x_2 = 0.04$

$$y_3 = y_2 + hf(x_2, y_2) = 1.0408 + 0.02 \times 1.0808 = 1.0624$$

The calculations can be put in table as below:

x_n	y_n	y'_n	hy'_n
0	1.0000	1.0000	0.0200
0.02	1.0200	1.0400	0.0208
0.04	1.0408	1.0808	0.0216
0.06	1.0624	1.1224	0.0224
0.08	1.0848	1.1648	0.0233
0.1	1.1081		

The analytical solution for the above DE is $y = 2e^x - x - 1 = 1.1103$,
$ey = 0.0022$

Example 2: Use Euler's method to approximate the solution to:

$$y' = y - x^2 + 1 \quad , \quad 0 \le x \le 2, \quad h = 0.5, \quad y(0) = 0.5$$

at $x = 2$.

Solution:

$x_0 = 0\,, \quad y_0 = y(0) = 0.5$

$$y_1 = y_0 + 0.5 \times (y_0 - 0^2 + 1) = 0.5 + 0.5 \times (0.5 - 0^2 + 1) = 1.25$$

$x_1 = 0.5$

$$y_2 = y_1 + 0.5 \times (y_1 - 0.5^2 + 1) = 1.25 + 0.5 \times (1.25 - 0.5^2 + 1) = 2.25$$

$x_2 = 1$

$$y_3 = y_2 + 0.5 \times (y_2 - 1^2 + 1) = 2.25 + 0.5 \times (2.25 - 1^2 + 1) = 3.375$$

$x_3 = 1.5$

$$y_4 = y_3 + 0.5 \times (y_3 - 1.5^2 + 1) = 3.375 + 0.5 \times (3.375 - 1.5^2 + 1) = 4.4375$$

TAYLOR METHOD

The formula of Taylor series is:

$$y(x+h) = y(x) + \frac{h}{1!}y'(x) + \frac{h^2}{2!}y''(x) + \frac{h^3}{3!}y'''(x) + \frac{h^4}{4!}y^{(4)} + \cdots$$

Where $h = x_{i+1} - x_i$.

To solve the following IVP:

$$\frac{dy}{dx} = f(x,y) \qquad a \le x \le b, \quad y(a) = \alpha$$

We find the successive derivatives of the analytic function f(x,y) with respect to the variables x, y. We substitute in Taylor's series to obtain:

$$y_1 = y(x_1) = y(x_0) + \frac{h}{1!}y'(x_0) + \frac{h^2}{2!}y''(x_0) + \frac{h^3}{3!}y'''(x_0) + \cdots$$

Which is called expanding about x_0. And so we continue expanding about x_i until we get the approximate solution $y_{n+1}(x)$ for the given differential equation.

$$\boldsymbol{y_{i+1} = y(x_i) = y(x_i) + \frac{h}{1!}y'(x_i) + \frac{h^2}{2!}y''(x_i) + \frac{h^3}{3!}y'''(x_i) + R}$$

R: is the remainder term which represent the truncated terms from Taylor series.

Example 3: Find the solution to the following DE using five terms of Taylor series:

$$\frac{dy}{dx} = x.y \ , \quad y(0) = 1, \quad 0 \le x \le 0.2, \quad h = 0.1$$

Solution:

The first step:

$y'(0) = (0)(1) = 0$

$y''(x) = y + xy' \Rightarrow y''(0) = 1 + (0) = 1$

$y'''(x) = y' + y' + xy'' = 2y' + xy''$

$y'''(0) = 2(0) + (0)(1) = 0$

$y^{(4)}(x) = 2y'' + y'' + xy''' = 3y'' + xy'''$

$y^{(4)}(0) = 3(1) + (0)(0) = 3$

Substituting in Taylor series:

$$y_1 = y(x_1) = y(x_0) + \frac{h}{1!}y'(x_0) + \frac{h^2}{2!}y''(x_0) + \frac{h^3}{3!}y'''(x_0) + \frac{h^4}{4!}y^{(4)}(x_0)$$

$$y_1 = 1 + (0.1)(0) + \frac{(0.1)^2}{2}(1) + \frac{(0.1)^3}{6}(0) + \frac{(0.1)^4}{24}(3)$$

$$y_1 = 1.005 = y(0.1)$$

The second step:

$y_1'(x_1) = y'(0.1) = x_1 y_1 = (0.1)(1.005) = 0.1005$

$y''(x_1) = y_1 + x_1 y_1' = (1.005) + (0.1)(0.1005) = 1.015$

$y'''(x_1) = 2y_1' + x_1 y_1'' = 2(0.1005) + (0.1)(1.015) = 0.3025$

$y^{(4)}(x_1) = 2y_1'' + x_1 y_1''' = 3(1.015) + (0.1)(0.302) = 3.07521$

Substituting in Taylor series:

$$y_2 = y(x_2) = y(x_1) + \frac{h}{1!}y'(x_1) + \frac{h^2}{2!}y''(x_1) + \frac{h^3}{3!}y'''(x_1) + \frac{h^4}{4!}y^{(4)}(x_1)$$

$$y_2 = 1.005 + (0.1)(0.1005) + \frac{(0.1)^2}{2}(1.015) + \frac{(0.1)^2}{6}(0.3025) + \frac{(0.1)^4}{24}(3.0752)$$

$$\boldsymbol{y_2 = 1.020188}$$

The analytical solution for the problem

$$y' = x.y \quad \Rightarrow \frac{dy}{dx} = x.y \;\Rightarrow \frac{dy}{y} = xdx$$

$$\int_{y=1}^{y} \frac{dy}{y} = \int_{x=0}^{x} xdx$$

$$[Ln]_1^y = [\frac{x^2}{2}]_0^x \quad \Rightarrow \quad Ln\, y = \frac{x^2}{2} \quad \Rightarrow \qquad y = e^{\frac{x^2}{2}}$$

Substitute (0.2) we obtain

$$y = e^{\frac{(0.2)^2}{2}} = e^{0.02} = \mathbf{1.0202}$$

The analytical result is approximately the same as the numerical result.

The error committed in solving the differential equation numerically can be estimated. When we satisfied with **k** terms of the Taylor series, the error is given by the following relationship:

$$\boldsymbol{e(x) = \frac{h^{n+1}}{(k+1)!}y^{(k+1)}(x); \quad x_n \leq x \leq x_n + h}$$

Example 4: To estimate the error in the example 3, we find $f^{(5)}$

$$f^{(5)} = 3y''' + y''' + xy^{(4)} = 4y''' + xy^{(4)}$$

$$f^{(5)} = (4)(0) + (0.2)(3) = 0.6$$

$$e(x) = \frac{h^{n+1}}{(k+1)!}y^{(k+1)}(x) = \frac{(0.1)^5}{(5)!}(0.6)(0.2) = 0.00000001$$