

Composite Functions

Composition is another method for combining functions.

DEFINITION Composition of Functions

If f and g are functions, the **composite** function $f \circ g$ (“ f composed with g ”) is defined by

$$(f \circ g)(x) = f(g(x)).$$

The domain of $f \circ g$ consists of the numbers x in the domain of g for which $g(x)$ lies in the domain of f .

The definition says that $f \circ g$ can be formed when the range of g lies in the domain of f . To find $(f \circ g)(x)$, *first* find $g(x)$ and *second* find $f(g(x))$. Figure 1.36 pictures $f \circ g$ as a machine diagram and Figure 1.37 shows the composite as an arrow diagram.

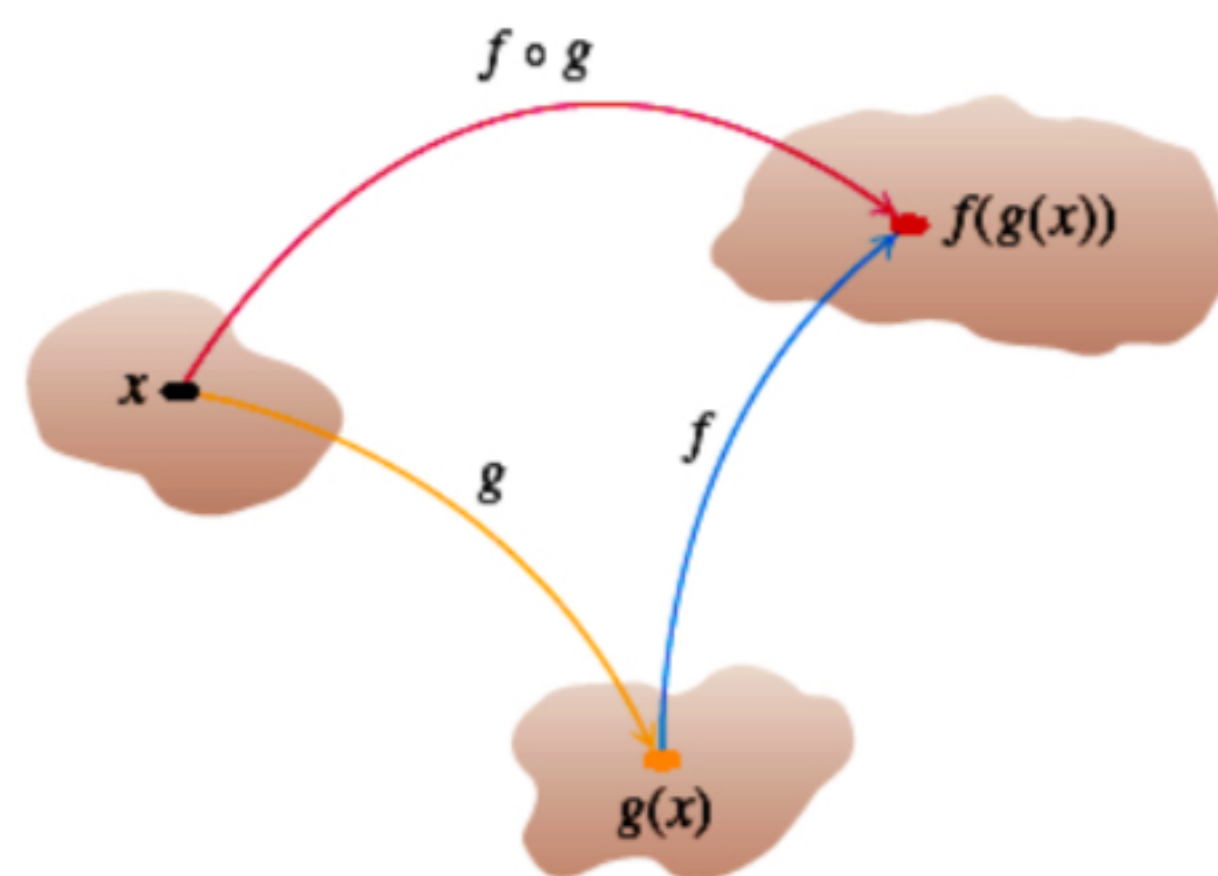
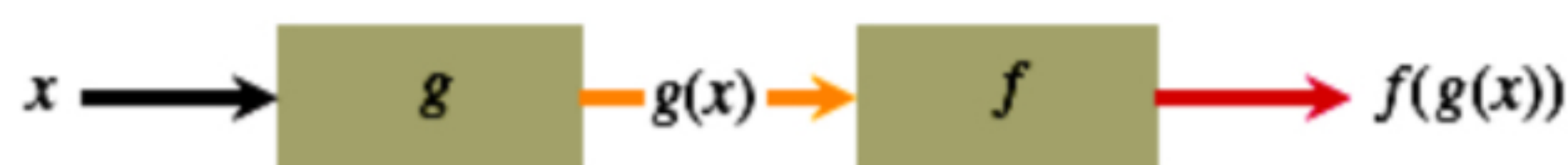


FIGURE 1.36 Two functions can be composed at x whenever the value of one function at x lies in the domain of the other. The composite is denoted by $f \circ g$.

FIGURE 1.37 Arrow diagram for $f \circ g$.

EXAMPLE 2 Viewing a Function as a Composite

The function $y = \sqrt{1 - x^2}$ can be thought of as first calculating $1 - x^2$ and then taking the square root of the result. The function y is the composite of the function $g(x) = 1 - x^2$ and the function $f(x) = \sqrt{x}$. Notice that $1 - x^2$ cannot be negative. The domain of the composite is $[-1, 1]$. ■

To evaluate the composite function $g \circ f$ (when defined), we reverse the order, finding $f(x)$ first and then $g(f(x))$. The domain of $g \circ f$ is the set of numbers x in the domain of f such that $f(x)$ lies in the domain of g .

The functions $f \circ g$ and $g \circ f$ are usually quite different.

EXAMPLE 3 Finding Formulas for Composites

If $f(x) = \sqrt{x}$ and $g(x) = x + 1$, find

- (a) $(f \circ g)(x)$ (b) $(g \circ f)(x)$ (c) $(f \circ f)(x)$ (d) $(g \circ g)(x)$.

Solution

Composite	Domain
(a) $(f \circ g)(x) = f(g(x)) = \sqrt{g(x)} = \sqrt{x + 1}$	$[-1, \infty)$
(b) $(g \circ f)(x) = g(f(x)) = f(x) + 1 = \sqrt{x} + 1$	$[0, \infty)$
(c) $(f \circ f)(x) = f(f(x)) = \sqrt{f(x)} = \sqrt{\sqrt{x}} = x^{1/4}$	$[0, \infty)$
(d) $(g \circ g)(x) = g(g(x)) = g(x) + 1 = (x + 1) + 1 = x + 2$	$(-\infty, \infty)$

To see why the domain of $f \circ g$ is $[-1, \infty)$, notice that $g(x) = x + 1$ is defined for all real x but belongs to the domain of f only if $x + 1 \geq 0$, that is to say, when $x \geq -1$. ■

Notice that if $f(x) = x^2$ and $g(x) = \sqrt{x}$, then $(f \circ g)(x) = (\sqrt{x})^2 = x$. However, the domain of $f \circ g$ is $[0, \infty)$, not $(-\infty, \infty)$.

Shifting a Graph of a Function

To shift the graph of a function $y = f(x)$ straight up, add a positive constant to the right-hand side of the formula $y = f(x)$.

To shift the graph of a function $y = f(x)$ straight down, add a negative constant to the right-hand side of the formula $y = f(x)$.

To shift the graph of $y = f(x)$ to the left, add a positive constant to x . To shift the graph of $y = f(x)$ to the right, add a negative constant to x .

Shift Formulas

Vertical Shifts

$y = f(x) + k$ Shifts the graph of f *up* k units if $k > 0$
Shifts it *down* $|k|$ units if $k < 0$

Horizontal Shifts

$y = f(x + h)$ Shifts the graph of f *left* h units if $h > 0$
Shifts it *right* $|h|$ units if $h < 0$

EXAMPLE 4 Shifting a Graph

- (a) Adding 1 to the right-hand side of the formula $y = x^2$ to get $y = x^2 + 1$ shifts the graph up 1 unit (Figure 1.38).
- (b) Adding -2 to the right-hand side of the formula $y = x^2$ to get $y = x^2 - 2$ shifts the graph down 2 units (Figure 1.38).
- (c) Adding 3 to x in $y = x^2$ to get $y = (x + 3)^2$ shifts the graph 3 units to the left (Figure 1.39).
- (d) Adding -2 to x in $y = |x|$, and then adding -1 to the result, gives $y = |x - 2| - 1$ and shifts the graph 2 units to the right and 1 unit down (Figure 1.40).

FIGURE 1.38 To shift the graph of $f(x) = x^2$ up (or down), we add positive (or negative) constants to the formula for f (Example 4a and b).

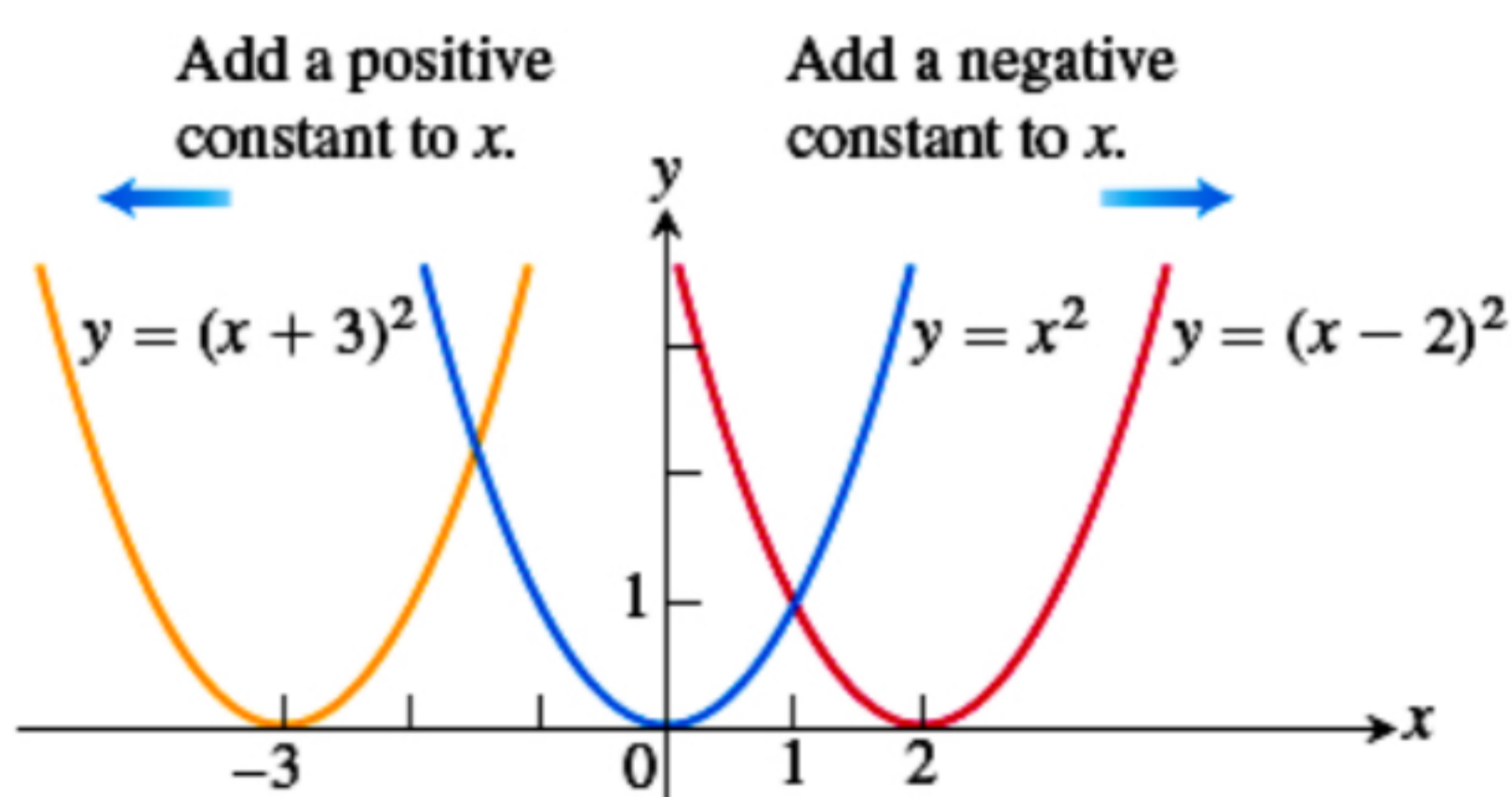
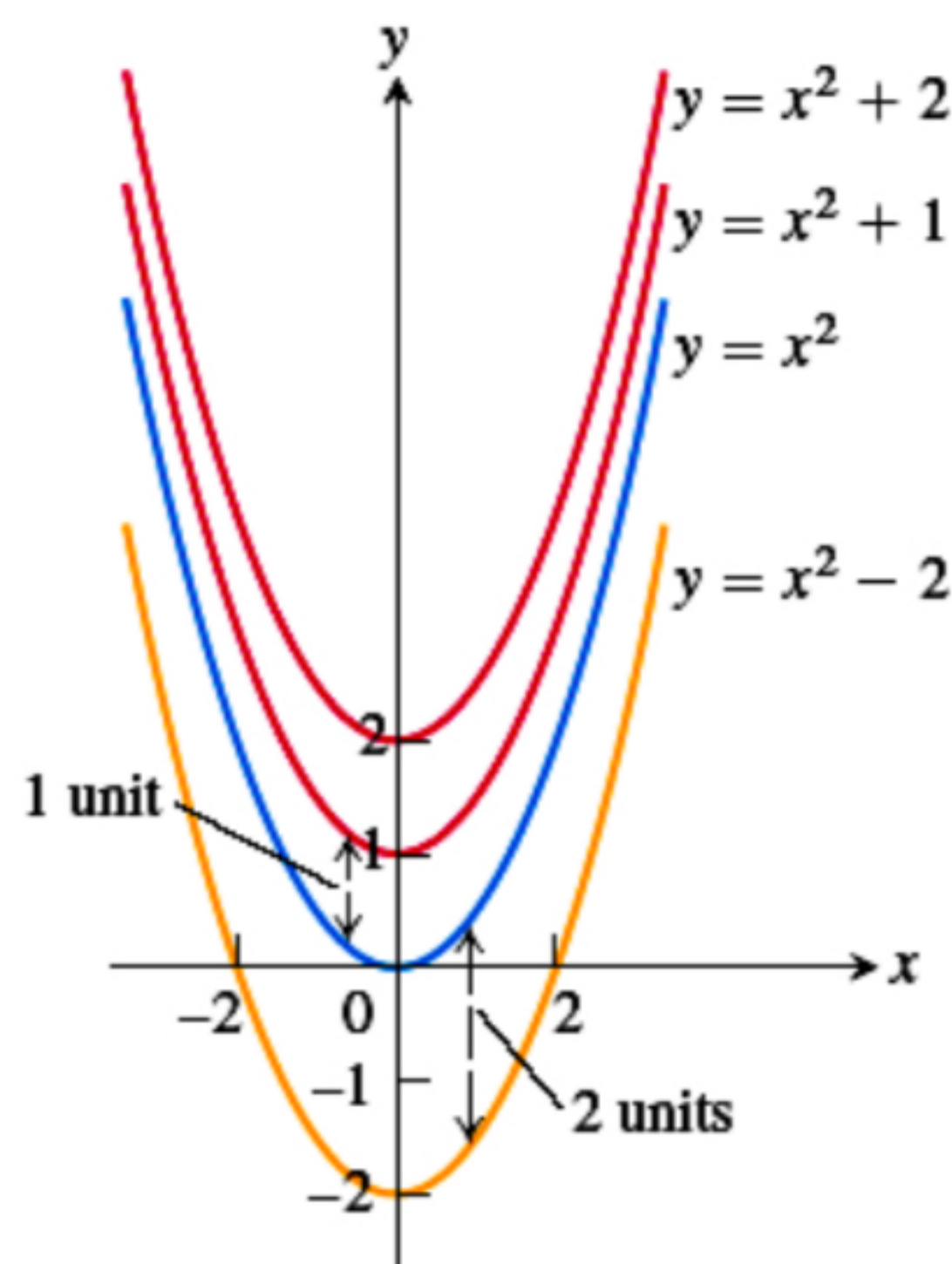


FIGURE 1.39 To shift the graph of $y = x^2$ to the left, we add a positive constant to x . To shift the graph to the right, we add a negative constant to x (Example 4c).

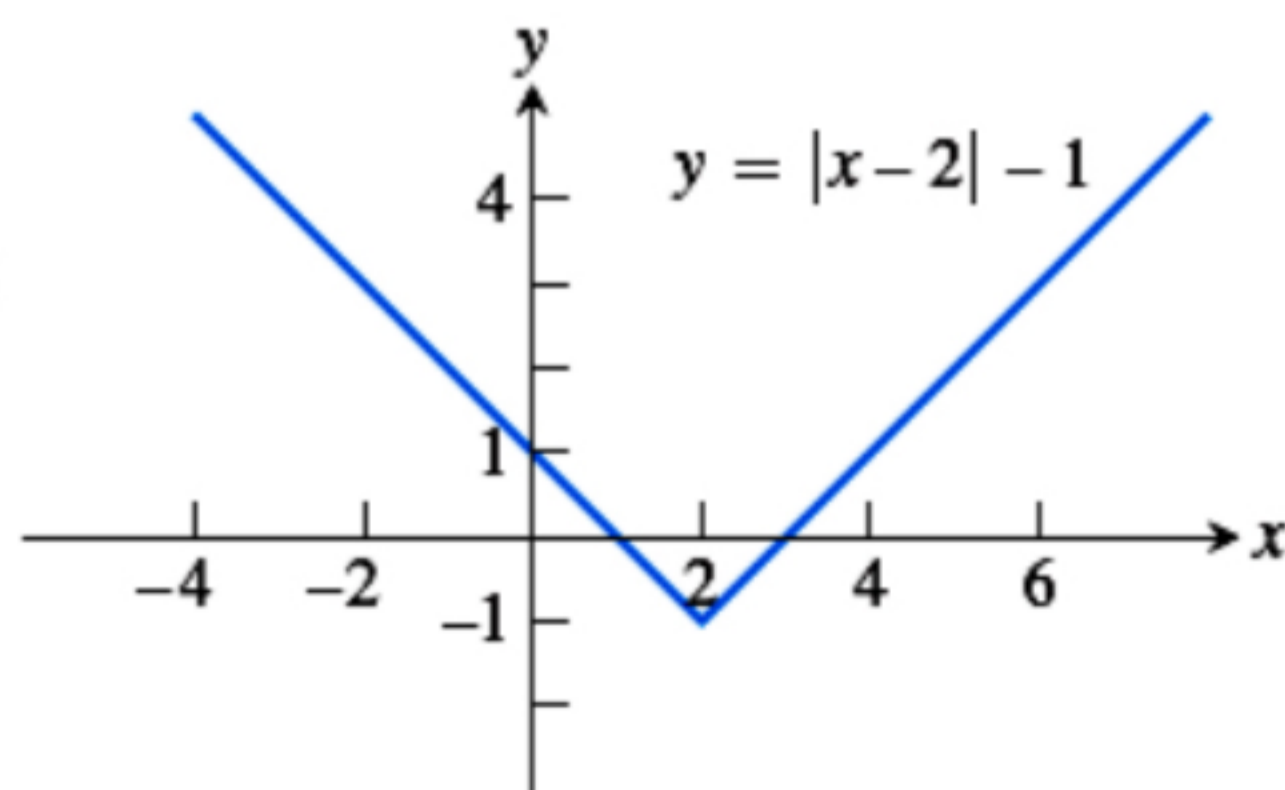


FIGURE 1.40 Shifting the graph of $y = |x|$ 2 units to the right and 1 unit down (Example 4d).

EXERCISES 1.3

Functions

In Exercises 1–6, find the domain and range of each function.

1. $f(x) = 1 + x^2$

2. $f(x) = 1 - \sqrt{x}$

3. $F(t) = \frac{1}{\sqrt{t}}$

4. $F(t) = \frac{1}{1 + \sqrt{t}}$

5. $g(z) = \sqrt{4 - z^2}$

6. $g(z) = \frac{1}{\sqrt{4 - z^2}}$

9. Consider the function $y = \sqrt{(1/x) - 1}$

a. Can x be negative?

b. Can $x = 0$?

c. Can x be greater than 1?

d. What is the domain of the function?

10. Consider the function $y = \sqrt{2 - \sqrt{x}}$.

a. Can x be negative?

b. Can \sqrt{x} be greater than 2?

c. What is the domain of the function?

Functions and Graphs

Find the domain and graph the functions in Exercises 15–20.

15. $f(x) = 5 - 2x$

16. $f(x) = 1 - 2x - x^2$

17. $g(x) = \sqrt{|x|}$

18. $g(x) = \sqrt{-x}$

19. $F(t) = t/|t|$

20. $G(t) = 1/|t|$

21. Graph the following equations and explain why they are not graphs of functions of x .

a. $|y| = x$

b. $y^2 = x^2$

22. Graph the following equations and explain why they are not graphs of functions of x .

a. $|x| + |y| = 1$

b. $|x + y| = 1$

Piecewise-Defined Functions

Graph the functions in Exercises 23–26.

23. $f(x) = \begin{cases} x, & 0 \leq x \leq 1 \\ 2 - x, & 1 < x \leq 2 \end{cases}$

25. $F(x) = \begin{cases} 3 - x, & x \leq 1 \\ 2x, & x > 1 \end{cases}$

24. $g(x) = \begin{cases} 1 - x, & 0 \leq x \leq 1 \\ 2 - x, & 1 < x \leq 2 \end{cases}$

26. $G(x) = \begin{cases} 1/x, & x < 0 \\ x, & 0 \leq x \end{cases}$

EXERCISES 1.4

Even and Odd Functions

In Exercises 19–30, say whether the function is even, odd, or neither. Give reasons for your answer.

19. $f(x) = 3$

21. $f(x) = x^2 + 1$

23. $g(x) = x^3 + x$

25. $g(x) = \frac{1}{x^2 - 1}$

27. $h(t) = \frac{1}{t - 1}$

29. $h(t) = 2t + 1$

20. $f(x) = x^{-5}$

22. $f(x) = x^2 + x$

24. $g(x) = x^4 + 3x^2 - 1$

26. $g(x) = \frac{x}{x^2 - 1}$

28. $h(t) = |t^3|$

30. $h(t) = 2|t| + 1$