

Limits And Continuity

2.1 Calculating Limits Using the Limit Laws

Limits of Function Values

Our examples have suggested the limit idea. Let's begin with an informal definition of limit, postponing the precise definition until we've gained more insight.

Let $f(x)$ be defined on an open interval about x_0 , *except possibly at x_0 itself*. If $f(x)$ gets arbitrarily close to L (as close to L as we like) for all x sufficiently close to x_0 , we say that f approaches the **limit** L as x approaches x_0 , and we write

$$\lim_{x \rightarrow x_0} f(x) = L,$$

which is read "the limit of $f(x)$ as x approaches x_0 is L ". Essentially, the definition says that the values of $f(x)$ are close to the number L whenever x is close to x_0 (on either side of x_0). This definition is "informal" because phrases like *arbitrarily close* and *sufficiently close* are imprecise; their meaning depends on the context. To a machinist manufacturing a piston, *close* may mean *within a few thousandths of an inch*.

EXAMPLE 1 Behavior of a Function Near a Point

How does the function

$$f(x) = \frac{x^2 - 1}{x - 1}$$

behave near $x = 1$?

Solution The given formula defines f for all real numbers x except $x = 1$ (we cannot divide by zero). For any $x \neq 1$, we can simplify the formula by factoring the numerator and canceling common factors:

$$f(x) = \frac{(x - 1)(x + 1)}{x - 1} = x + 1 \quad \text{for } x \neq 1.$$

The graph of f is thus the line $y = x + 1$ with the point $(1, 2)$ *removed*. This removed point is shown as a “hole” in Figure 2.1 Even though $f(1)$ is not defined, it is clear that we can make the value of $f(x)$ *as close as we want* to 2 by choosing x *close enough* to 1

The closer x gets to 1, the closer $f(x) = (x^2 - 1)/(x - 1)$ seems to get to 2

Values of x below and above 1	$f(x) = \frac{x^2 - 1}{x - 1} = x + 1, \quad x \neq 1$
0.9	1.9
1.1	2.1
0.99	1.99
1.01	2.01
0.999	1.999
1.001	2.001
0.999999	1.999999
1.000001	2.000001

We say that $f(x)$ approaches the *limit* 2 as x approaches 1, and write

$$\lim_{x \rightarrow 1} f(x) = 2, \quad \text{or} \quad \lim_{x \rightarrow 1} \frac{x^2 - 1}{x - 1} = 2. \quad \blacksquare$$

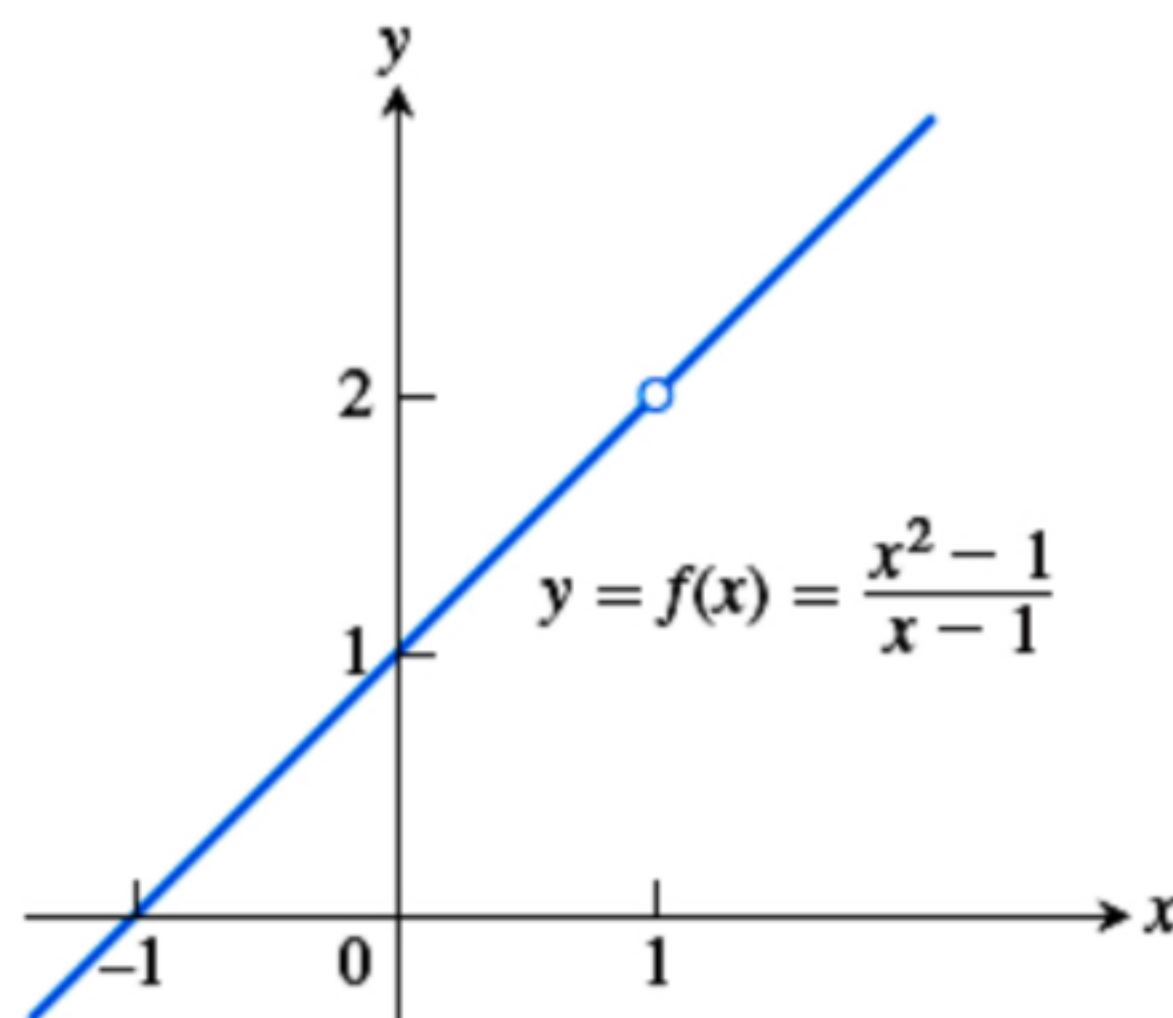
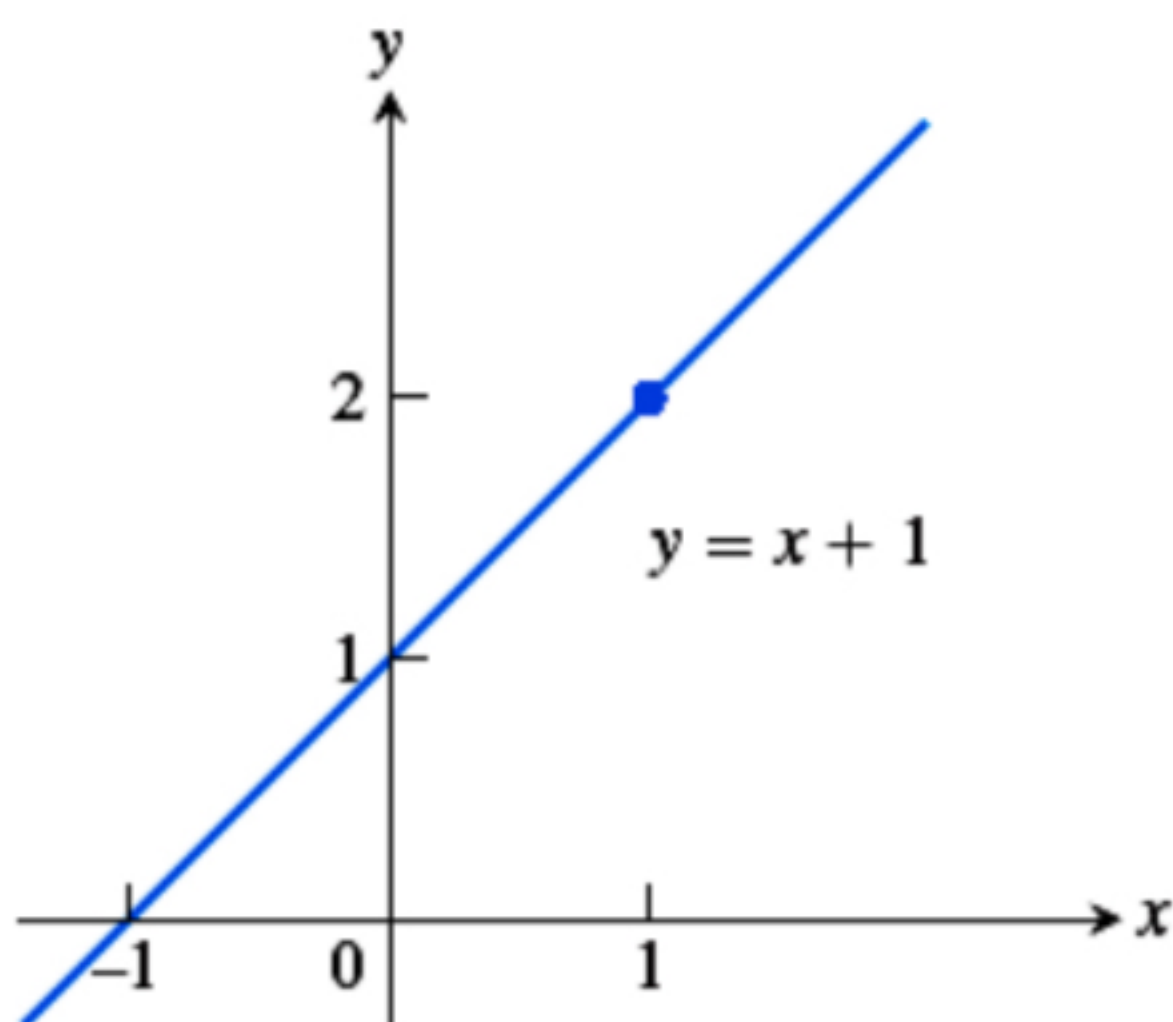


FIGURE 2.1 The graph of f is identical with the line $y = x + 1$ except at $x = 1$, where f is not defined (Example 1).

EXAMPLE 2 The Limit Value Does Not Depend on How the Function Is Defined at x_0

The function f in Figure 2.2 has limit 2 as $x \rightarrow 1$ even though f is not defined at $x = 1$. The function g has limit 2 as $x \rightarrow 1$ even though $2 \neq g(1)$. The function h is the only one

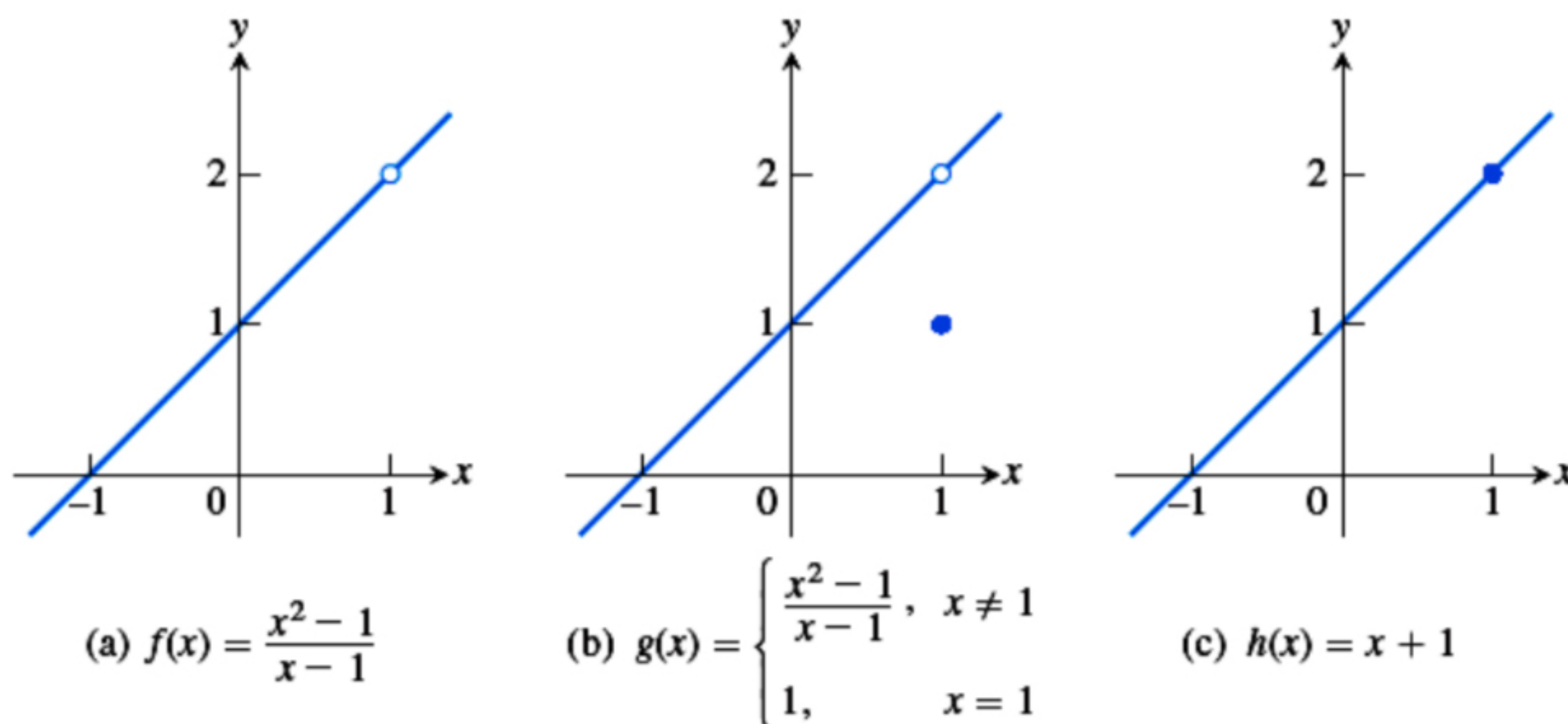


FIGURE 2.2 The limits of $f(x)$, $g(x)$, and $h(x)$ all equal 2 as x approaches 1. However, only $h(x)$ has the same function value as its limit at $x = 1$ (Example 2).

whose limit as $x \rightarrow 1$ equals its value at $x = 1$. For h , we have $\lim_{x \rightarrow 1} h(x) = h(1)$. This equality of limit and function value is special, ■

Sometimes $\lim_{x \rightarrow x_0} f(x)$ can be evaluated by calculating $f(x_0)$. This holds, for example, whenever $f(x)$ is an algebraic combination of polynomials and trigonometric functions for which $f(x_0)$ is defined.

EXAMPLE 3 Finding Limits by Calculating $f(x_0)$

- (a) $\lim_{x \rightarrow 2} (4) = 4$
- (b) $\lim_{x \rightarrow -13} (4) = 4$
- (c) $\lim_{x \rightarrow 3} x = 3$
- (d) $\lim_{x \rightarrow 2} (5x - 3) = 10 - 3 = 7$
- (e) $\lim_{x \rightarrow -2} \frac{3x + 4}{x + 5} = \frac{-6 + 4}{-2 + 5} = -\frac{2}{3}$

EXAMPLE 4 The Identity and Constant Functions Have Limits at Every Point

(a) If f is the **identity function** $f(x) = x$, then for any value of x_0 (Figure 2.3a),

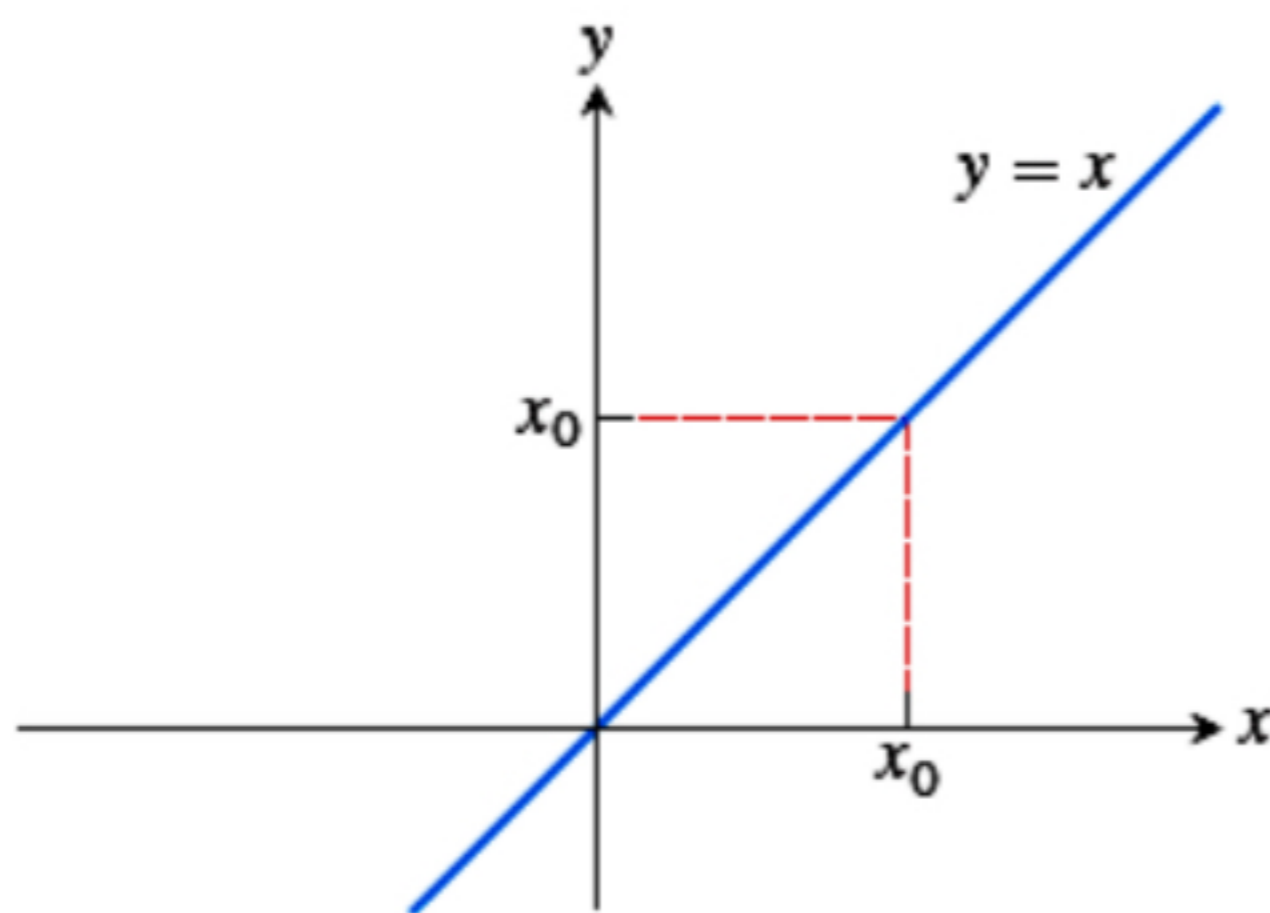
$$\lim_{x \rightarrow x_0} f(x) = \lim_{x \rightarrow x_0} x = x_0.$$

(b) If f is the **constant function** $f(x) = k$ (function with the constant value k), then for any value of x_0 (Figure 2.3b),

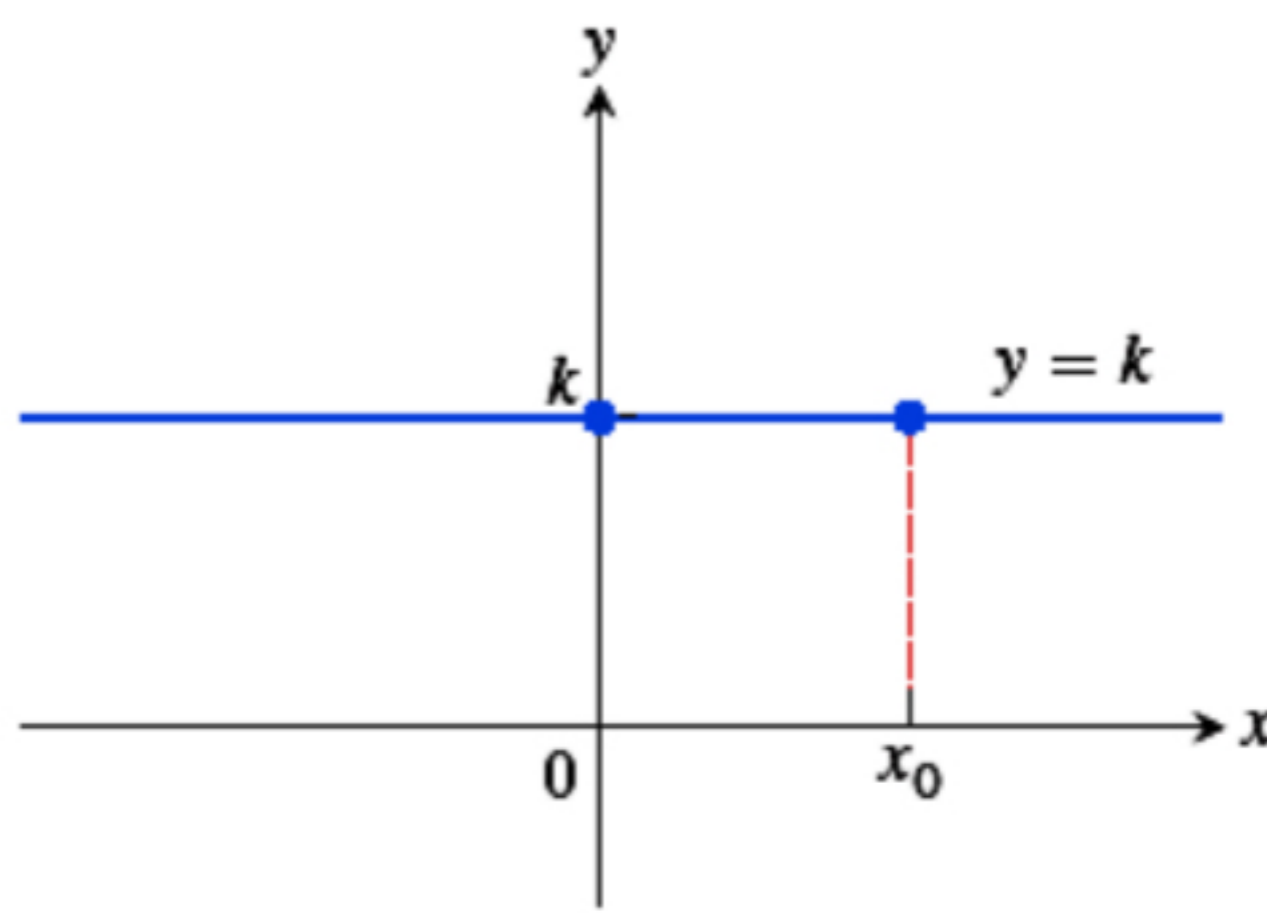
$$\lim_{x \rightarrow x_0} f(x) = \lim_{x \rightarrow x_0} k = k.$$

For instance,

$$\lim_{x \rightarrow 3} x = 3 \quad \text{and} \quad \lim_{x \rightarrow -7} (4) = \lim_{x \rightarrow 2} (4) = 4.$$



(a) Identity function



(b) Constant function

FIGURE 2.3 The functions in Example 8.