

EXAMPLE 6 Using the Limit Laws

Use the observations $\lim_{x \rightarrow c} k = k$ and $\lim_{x \rightarrow c} x = c$ and the properties of limits to find the following limits.

and the

$$(a) \lim_{x \rightarrow c} (x^3 + 4x^2 - 3) \quad (b) \lim_{x \rightarrow c} \frac{x^4 + x^2 - 1}{x^2 + 5} \quad (c) \lim_{x \rightarrow -2} \sqrt{4x^2 - 3}$$

Solution

$$(a) \lim_{x \rightarrow c} (x^3 + 4x^2 - 3) = \lim_{x \rightarrow c} x^3 + \lim_{x \rightarrow c} 4x^2 - \lim_{x \rightarrow c} 3 \quad \text{Sum and Difference Rules}$$
$$= c^3 + 4c^2 - 3 \quad \text{Product and Multiple Rules}$$

$$(b) \lim_{x \rightarrow c} \frac{x^4 + x^2 - 1}{x^2 + 5} = \frac{\lim_{x \rightarrow c} (x^4 + x^2 - 1)}{\lim_{x \rightarrow c} (x^2 + 5)} \quad \text{Quotient Rule}$$
$$= \frac{\lim_{x \rightarrow c} x^4 + \lim_{x \rightarrow c} x^2 - \lim_{x \rightarrow c} 1}{\lim_{x \rightarrow c} x^2 + \lim_{x \rightarrow c} 5} \quad \text{Sum and Difference Rules}$$
$$= \frac{c^4 + c^2 - 1}{c^2 + 5} \quad \text{Power or Product Rule}$$

$$(c) \lim_{x \rightarrow -2} \sqrt{4x^2 - 3} = \sqrt{\lim_{x \rightarrow -2} (4x^2 - 3)} \quad \text{Power Rule with } r/s = 1/2$$
$$= \sqrt{\lim_{x \rightarrow -2} 4x^2 - \lim_{x \rightarrow -2} 3} \quad \text{Difference Rule}$$
$$= \sqrt{4(-2)^2 - 3} \quad \text{Product and Multiple Rules}$$
$$= \sqrt{16 - 3}$$
$$= \sqrt{13}$$

THEOREM 2 Limits of Polynomials Can Be Found by Substitution

If $P(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_0$, then

$$\lim_{x \rightarrow c} P(x) = P(c) = a_n c^n + a_{n-1} c^{n-1} + \cdots + a_0.$$

THEOREM 3 Limits of Rational Functions Can Be Found by Substitution
If the Limit of the Denominator Is Not Zero

If $P(x)$ and $Q(x)$ are polynomials and $Q(c) \neq 0$, then

$$\lim_{x \rightarrow c} \frac{P(x)}{Q(x)} = \frac{P(c)}{Q(c)}.$$

EXAMPLE 7 Limit of a Rational Function

$$\lim_{x \rightarrow -1} \frac{x^3 + 4x^2 - 3}{x^2 + 5} = \frac{(-1)^3 + 4(-1)^2 - 3}{(-1)^2 + 5} = \frac{0}{6} = 0$$

This result is similar to the second limit in Example 1 with $c = -1$, now done in one step. ■

Eliminating Zero Denominators Algebraically

Theorem 3 applies only if the denominator of the rational function is not zero at the limit point c . If the denominator is zero, canceling common factors in the numerator and denominator may reduce the fraction to one whose denominator is no longer zero at c . If this happens, we can find the limit by substitution in the simplified fraction.

EXAMPLE 8 Canceling a Common Factor

Evaluate

$$\lim_{x \rightarrow 1} \frac{x^2 + x - 2}{x^2 - x}.$$

Solution We cannot substitute $x = 1$ because it makes the denominator zero. We test the numerator to see if it, too, is zero at $x = 1$. It is, so it has a factor of $(x - 1)$ in common with the denominator. Canceling the $(x - 1)$'s gives a simpler fraction with the same values as the original for $x \neq 1$:

$$\frac{x^2 + x - 2}{x^2 - x} = \frac{(x - 1)(x + 2)}{x(x - 1)} = \frac{x + 2}{x}, \quad \text{if } x \neq 1.$$

Using the simpler fraction, we find the limit of these values as $x \rightarrow 1$ by substitution:

$$\lim_{x \rightarrow 1} \frac{x^2 + x - 2}{x^2 - x} = \lim_{x \rightarrow 1} \frac{x + 2}{x} = \frac{1 + 2}{1} = 3.$$

See Figure 2.5 ■

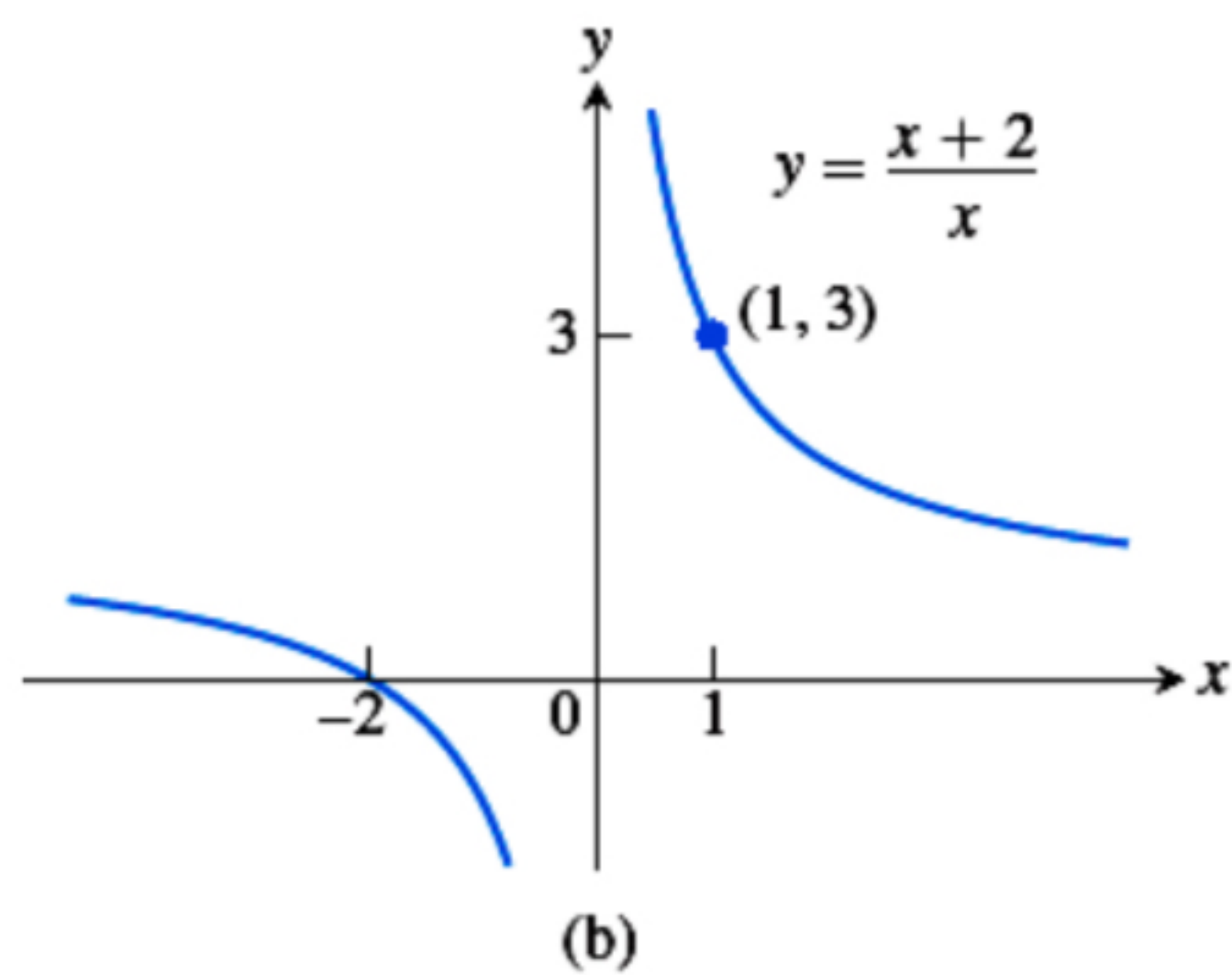
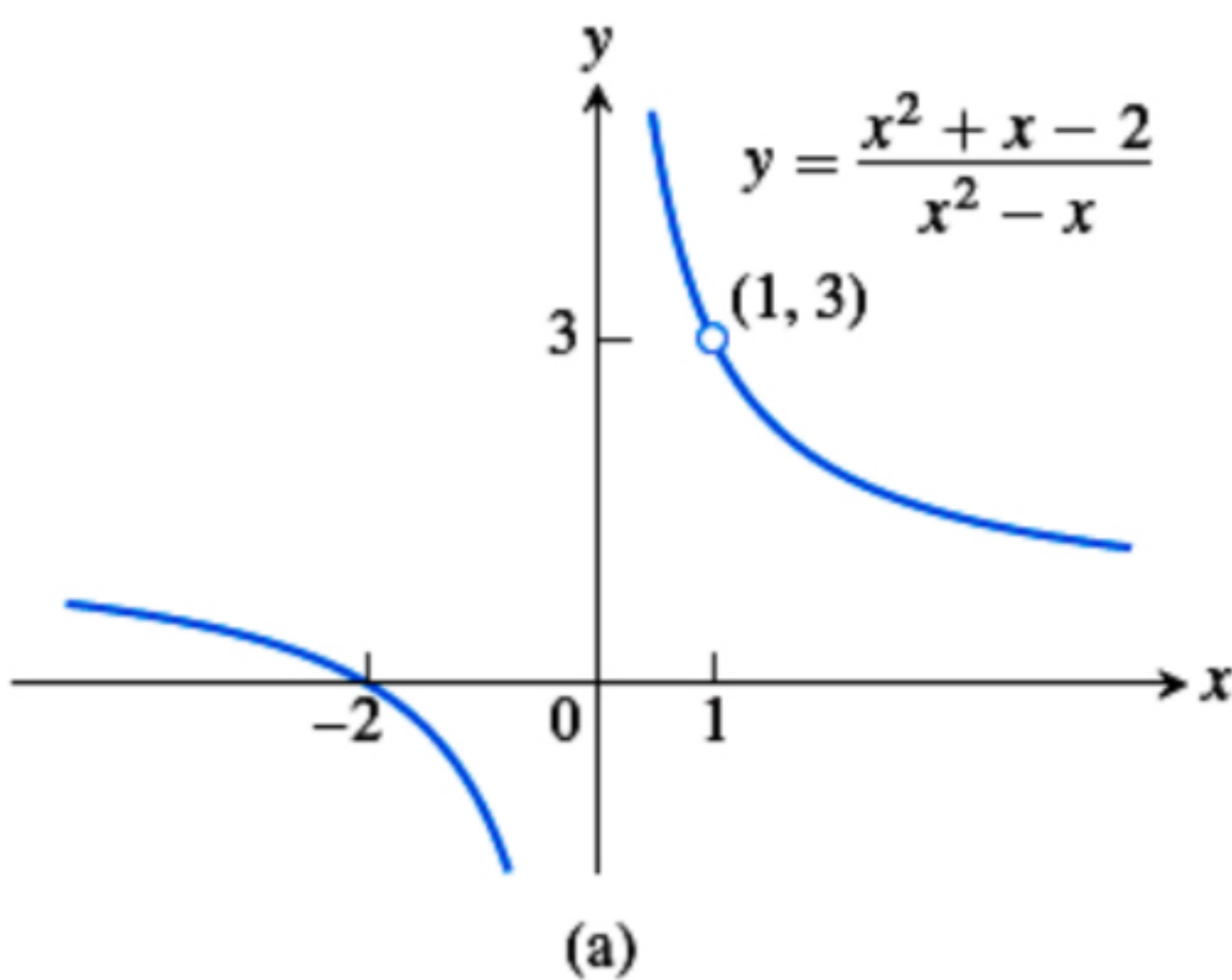


FIGURE 2.5 The graph of $f(x) = (x^2 + x - 2)/(x^2 - x)$ in part (a) is the same as the graph of $g(x) = (x + 2)/x$ in part (b) except at $x = 1$, where f is undefined. The functions have the same limit as $x \rightarrow 1$ (Example 8).

EXAMPLE 9 Creating and Canceling a Common Factor

Evaluate

$$\lim_{x \rightarrow 0} \frac{\sqrt{x^2 + 100} - 10}{x^2}.$$

Solution

cannot substitute $x = 0$, and the numerator and denominator have no obvious common factors. We can create a common factor by multiplying both numerator and denominator by the expression $\sqrt{x^2 + 100} + 10$ (obtained by changing the sign after the square root). The preliminary algebra rationalizes the numerator:

$$\begin{aligned} \frac{\sqrt{x^2 + 100} - 10}{x^2} &= \frac{\sqrt{x^2 + 100} - 10}{x^2} \cdot \frac{\sqrt{x^2 + 100} + 10}{\sqrt{x^2 + 100} + 10} \\ &= \frac{x^2 + 100 - 100}{x^2(\sqrt{x^2 + 100} + 10)} \\ &= \frac{x^2}{x^2(\sqrt{x^2 + 100} + 10)} \end{aligned}$$

$$= \frac{1}{\sqrt{x^2 + 100} + 10}.$$

Cancel x^2 for $x \neq 0$

Therefore,

$$\begin{aligned}\lim_{x \rightarrow 0} \frac{\sqrt{x^2 + 100} - 10}{x^2} &= \lim_{x \rightarrow 0} \frac{1}{\sqrt{x^2 + 100} + 10} \\ &= \frac{1}{\sqrt{0^2 + 100} + 10} \\ &= \frac{1}{20} = 0.05.\end{aligned}$$

Denominator
not 0 at $x = 0$;
substitute