Derivatives of the Other Basic Trigonometric Functions

Because $\sin x$ and $\cos x$ are differentiable functions of x, the related functions

$$\tan x = \frac{\sin x}{\cos x}, \qquad \cot x = \frac{\cos x}{\sin x}, \qquad \sec x = \frac{1}{\cos x}, \qquad \text{and} \qquad \csc x = \frac{1}{\sin x}$$

are differentiable at every value of x at which they are defined. Their derivatives, calculated from the Quotient Rule, are given by the following formulas. Notice the negative signs in the derivative formulas for the cofunctions.

Derivatives of the Other Trigonometric Functions

$$\frac{d}{dx}(\tan x) = \sec^2 x$$

$$\frac{d}{dx}(\sec x) = \sec x \tan x$$

$$\frac{d}{dx}(\cot x) = -\csc^2 x$$

$$\frac{d}{dx}(\csc x) = -\csc x \cot x$$

EXAMPLE 5

Find $d(\tan x)/dx$.

Solution

$$\frac{d}{dx}(\tan x) = \frac{d}{dx}\left(\frac{\sin x}{\cos x}\right) = \frac{\cos x \frac{d}{dx}(\sin x) - \sin x \frac{d}{dx}(\cos x)}{\cos^2 x} \qquad \text{Quotient Rule}$$

$$= \frac{\cos x \cos x - \sin x(-\sin x)}{\cos^2 x}$$

$$= \frac{\cos^2 x + \sin^2 x}{\cos^2 x}$$

$$= \frac{1}{\cos^2 x} = \sec^2 x$$

■

EXAMPLE 6

Find y'' if $y = \sec x$.

Solution

$$y = \sec x$$

$$y' = \sec x \tan x$$

$$\begin{aligned} y'' &= \frac{d}{dx}(\sec x \tan x) \\ &= \sec x \frac{d}{dx}\left(\tan x\right) + \tan x \frac{d}{dx}\left(\sec x\right) \qquad \text{Product Rule} \\ &= \sec x(\sec^2 x) + \tan x(\sec x \tan x) \\ &= \sec^3 x + \sec x \tan^2 x \end{aligned}$$

EXAMPLE 7 Finding a Trigonometric Limit

$$\lim_{x\to 0} \frac{\sqrt{2+\sec x}}{\cos(\pi - \tan x)} = \frac{\sqrt{2+\sec 0}}{\cos(\pi - \tan 0)} = \frac{\sqrt{2+1}}{\cos(\pi - 0)} = \frac{\sqrt{3}}{-1} = -\sqrt{3}$$

EXERCISES 3.3

In Exercises 1–12, find dy/dx.

1. $y = -10x + 3\cos x$
2. $y = \frac{3}{x} + 5\sin x$
3. $y = \csc x - 4\sqrt{x} + 7$
4. $y = x^2 \cot x - \frac{1}{x^2}$
5. $y = (\sec x + \tan x)(\sec x - \tan x)$
6. $y = (\sin x + \cos x)\sec x$
7. $y = \frac{\cot x}{1 + \cot x}$
8. $y = \frac{\cos x}{1 + \sin x}$
9. $y = \frac{4}{\cos x} + \frac{1}{\tan x}$
10. $y = \frac{\cos x}{x} + \frac{x}{\cos x}$
11. $y = x^2 \sin x + 2x\cos x - 2\sin x$
12. $y = x^2 \cos x - 2x\sin x - 2\cos x$

In Exercises 13–16, find ds/dt.

13. $s = \tan t - t$
14. $s = t^2 - \sec t + 1$
15. $s = \frac{1 + \csc t}{1 - \csc t}$
16. $s = \frac{\sin t}{1 - \cos t}$

In Exercises 17–20, find $dr/d\theta$.

17. $r = 4 - \theta^2 \sin\theta$
18. $r = \theta\sin\theta + \cos\theta$
19. $r = \sec\theta\csc\theta$
20. $r = (1 + \sec\theta)\sin\theta$

In Exercises 21–24, find dp/dq.

21. $p = 5 + \frac{1}{\cot q}$

22. $p = (1 + \csc q)\cos q$

23. $p = \frac{\sin q + \cos q}{\cos q}$

24. $p = \frac{\tan q}{1 + \tan q}$

25. Find y'' if

a. $y = \csc x$.

b. $y = \sec x$.

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Is there a value of c that will make

$$f(x) = \begin{cases} \frac{\sin^2 3x}{x^2}, & x \neq 0 \\ c, & x = 0 \end{cases}$$

continuous at $x = 0$? Give reasons for your answer.

Trigonometric Limits

Find the limits in Exercises 27–32

27. $\lim_{x \to 2} \sin\left(\frac{1}{x} - \frac{1}{2}\right)$

28. $\lim_{x \to -\pi/6} \sqrt{1 + \cos(\pi \csc x)}$

29. $\lim_{x \to 0} \sec\left[\cos x + \pi \tan\left(\frac{\pi}{4 \sec x}\right) - 1\right]$

30. $\lim_{x \to 0} \sin\left(\frac{\pi + \tan x}{\tan x - 2 \sec x}\right)$

31. $\lim_{t \to 0} \tan\left(1 - \frac{\sin t}{t}\right)$

32. $\lim_{\theta \to 0} \cos\left(\frac{\pi\theta}{\sin\theta}\right)$

3.4 The Chain Rule and Parametric Equations

We know how to differentiate $y = f(u) = \sin u$ and $u = g(x) = x^2 - 4$, but how do we differentiate a composite like $F(x) = f(g(x)) = \sin(x^2 - 4)$? The differentiation formulas we have studied so far do not tell us how to calculate $F'(x)$. So how do we find the derivative of $F = f \circ g$? The answer is, with the Chain Rule, which says that the derivative of the composite of two differentiable functions is the product of their derivatives evaluated at appropriate points. The Chain Rule is one of the most important and widely used rules of differentiation. This section describes the rule and how to use it. We then apply the rule to describe curves in the plane and their tangent lines in another way.

EXAMPLE 1

The function $y = \frac{3}{2}x = \frac{1}{2}(3x)$ is the composite of the functions $y = \frac{1}{2}u$ and $u = 3x$. How are the derivatives of these functions related?

Solution We have

$$\frac{dy}{dx} = \frac{3}{2}, \qquad \frac{dy}{du} = \frac{1}{2}, \qquad \text{and} \qquad \frac{du}{dx} = 3.$$

Since $\frac{3}{2} = \frac{1}{2} \cdot 3$, we see that

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}.$$

EXAMPLE 2

The function

$$y = 9x^4 + 6x^2 + 1 = (3x^2 + 1)^2$$

is the composite of $y = u^2$ and $u = 3x^2 + 1$. Calculating derivatives, we see that

$$\begin{aligned} \frac{dy}{du} \cdot \frac{du}{dx} &= 2u \cdot 6x \\ &= 2(3x^2 + 1) \cdot 6x \\ &= 36x^3 + 12x. \end{aligned}$$

Calculating the derivative from the expanded formula, we get

$$\frac{dy}{dx} = \frac{d}{dx}\left(9x^4 + 6x^2 + 1\right) = 36x^3 + 12x.$$

Once again,

$$\frac{dy}{du} \cdot \frac{du}{dx} = \frac{dy}{dx}. \qquad \blacksquare$$

The derivative of the composite function $f(g(x))$ at x is the derivative of f at $g(x)$ times the derivative of g at x. This is known as the Chain Rule (Figure 7).

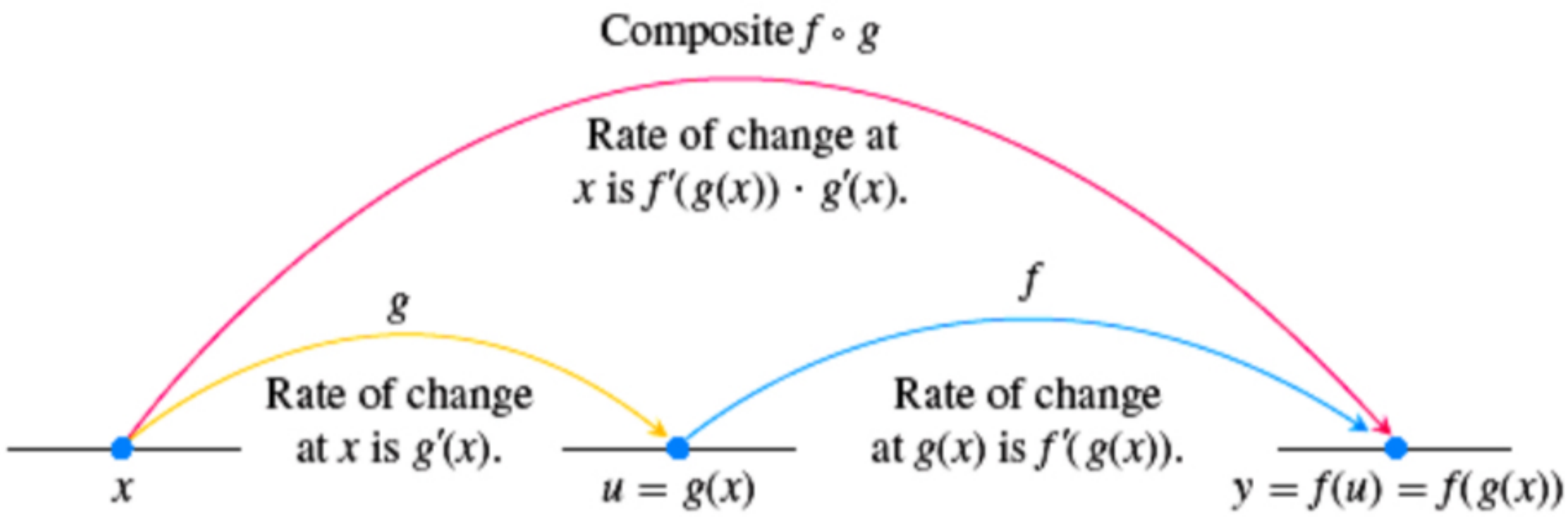

FIGURE 7 Rates of change multiply: The derivative of $f \circ g$ at x is the derivative of f at $g(x)$ times the derivative of g at x.

THEOREM **The Chain Rule**

If $f(u)$ is differentiable at the point $u = g(x)$ and $g(x)$ is differentiable at x, then the composite function $(f \circ g)(x) = f(g(x))$ is differentiable at x, and

$$(f \circ g)'(x) = f'(g(x)) \cdot g'(x).$$

In Leibniz's notation, if $y = f(u)$ and $u = g(x)$, then

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx},$$

where dy/du is evaluated at $u = g(x)$.

EXAMPLE 3 Applying the Chain Rule

An object moves along the x-axis so that its position at any time $t \geq 0$ is given by $x(t) = \cos(t^2 + 1)$. Find the velocity of the object as a function of t.

Solution We know that the velocity is dx/dt. In this instance, x is a composite function: $x = \cos(u)$ and $u = t^2 + 1$. We have

$$\frac{dx}{du} = -\sin(u) \qquad x = \cos(u)$$

$$\frac{du}{dt} = 2t. \qquad u = t^2 + 1$$

By the Chain Rule,

$$\begin{aligned}\frac{dx}{dt} &= \frac{dx}{du}\cdot\frac{du}{dt}\\ &= -\sin(u)\cdot 2t \qquad \frac{dx}{du} \text{ evaluated at } u\\ &= -\sin(t^2+1)\cdot 2t\\ &= -2t\sin(t^2+1).\end{aligned}$$

■

It sometimes helps to think about the Chain Rule this way: If $y = f(g(x))$, then

$$\frac{dy}{dx} = f'(g(x))\cdot g'(x).$$

In words, differentiate the "outside" function f and evaluate it at the "inside" function $g(x)$ left alone; then multiply by the derivative of the "inside function."

EXAMPLE 4

Differentiate $\sin(x^2 + x)$ with respect to x.

Solution

$$\frac{d}{dx}\sin\underbrace{(x^2+x)}_{\text{inside}} = \cos\underbrace{(x^2+x)}_{\substack{\text{inside}\\ \text{left alone}}}\cdot\underbrace{(2x+1)}_{\substack{\text{derivative of}\\ \text{the inside}}}$$

■

We sometimes have to use the Chain Rule two or more times to find a derivative. Here is an example.

EXAMPLE 5

Find the derivative of $g(t) = \tan(5 - \sin 2t)$.

Solution Notice here that the tangent is a function of $5 - \sin 2t$, whereas the sine is a function of $2t$, which is itself a function of t. Therefore, by the Chain Rule,

$$\begin{aligned}g'(t) &= \frac{d}{dt}\big(\tan(5-\sin 2t)\big)\\ &= \sec^2(5-\sin 2t)\cdot\frac{d}{dt}(5-\sin 2t) \qquad \text{Derivative of } \tan u \text{ with } u = 5-\sin 2t\\ &= \sec^2(5-\sin 2t)\cdot\left(0-\cos 2t\cdot\frac{d}{dt}(2t)\right) \qquad \text{Derivative of } 5-\sin u \text{ with } u = 2t\end{aligned}$$

$$= \sec^2(5 - \sin 2t) \cdot (-\cos 2t) \cdot 2$$
$$= -2(\cos 2t)\sec^2(5 - \sin 2t).$$ ■

The Chain Rule with Powers of a Function

If f is a differentiable function of u and if u is a differentiable function of x, then substituting $y = f(u)$ into the Chain Rule formula

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

leads to the formula

$$\frac{d}{dx} f(u) = f'(u)\frac{du}{dx}.$$

Here's an example of how it works: If n is a positive or negative integer and $f(u) = u^n$, the Power Rules (Rules 2 and 7) tell us that $f'(u) = nu^{n-1}$. If u is a differentiable function of x, then we can use the Chain Rule to extend this to the **Power Chain Rule**:

$$\frac{d}{dx} u^n = nu^{n-1}\frac{du}{dx}. \qquad \frac{d}{du}(u^n) = nu^{n-1}$$

EXAMPLE 6 Applying the Power Chain Rule

(a) $\frac{d}{dx}(5x^3 - x^4)^7 = 7(5x^3 - x^4)^6 \frac{d}{dx}(5x^3 - x^4)$ — Power Chain Rule with $u = 5x^3 - x^4, n = 7$

$$= 7(5x^3 - x^4)^6(5 \cdot 3x^2 - 4x^3)$$
$$= 7(5x^3 - x^4)^6(15x^2 - 4x^3)$$

(b) $\frac{d}{dx}\left(\frac{1}{3x - 2}\right) = \frac{d}{dx}(3x - 2)^{-1}$

$= -1(3x - 2)^{-2}\frac{d}{dx}(3x - 2)$ — Power Chain Rule with $u = 3x - 2, n = -1$

$$= -1(3x - 2)^{-2}(3)$$
$$= -\frac{3}{(3x - 2)^2}$$

In part (b) we could also have found the derivative with the Quotient Rule. ■

EXAMPLE 7 Finding Tangent Slopes

(a) Find the slope of the line tangent to the curve $y = \sin^5 x$ at the point where $x = \pi/3$.

(b) Show that the slope of every line tangent to the curve $y = 1/(1 - 2x)^3$ is positive.

Solution

(a) $\frac{dy}{dx} = 5\sin^4 x \cdot \frac{d}{dx}\sin x$ Power Chain Rule with $u = \sin x,\ n = 5$

$= 5\sin^4 x \cos x$

The tangent line has slope

$$\left.\frac{dy}{dx}\right|_{x=\pi/3} = 5\left(\frac{\sqrt{3}}{2}\right)^4\left(\frac{1}{2}\right) = \frac{45}{32}.$$

(b) $\frac{dy}{dx} = \frac{d}{dx}(1-2x)^{-3}$

$= -3(1-2x)^{-4} \cdot \frac{d}{dx}(1-2x)$ Power Chain Rule with $u = (1-2x),\ n = -3$

$= -3(1-2x)^{-4} \cdot (-2)$

$= \frac{6}{(1-2x)^4}$

At any point (x, y) on the curve, $x \neq 1/2$ and the slope of the tangent line is

$$\frac{dy}{dx} = \frac{6}{(1-2x)^4},$$

the quotient of two positive numbers. ■

Parametric Equations

Instead of describing a curve by expressing the y-coordinate of a point $P(x, y)$ on the curve as a function of x, it is sometimes more convenient to describe the curve by expressing *both* coordinates as functions of a third variable t. Figure 8 shows the path of a moving particle described by a pair of equations, $x = f(t)$ and $y = g(t)$. For studying motion, t usually denotes time. Equations like these are better than a Cartesian formula because they tell us the particle's position $(x, y) = (f(t), g(t))$ at any time t.

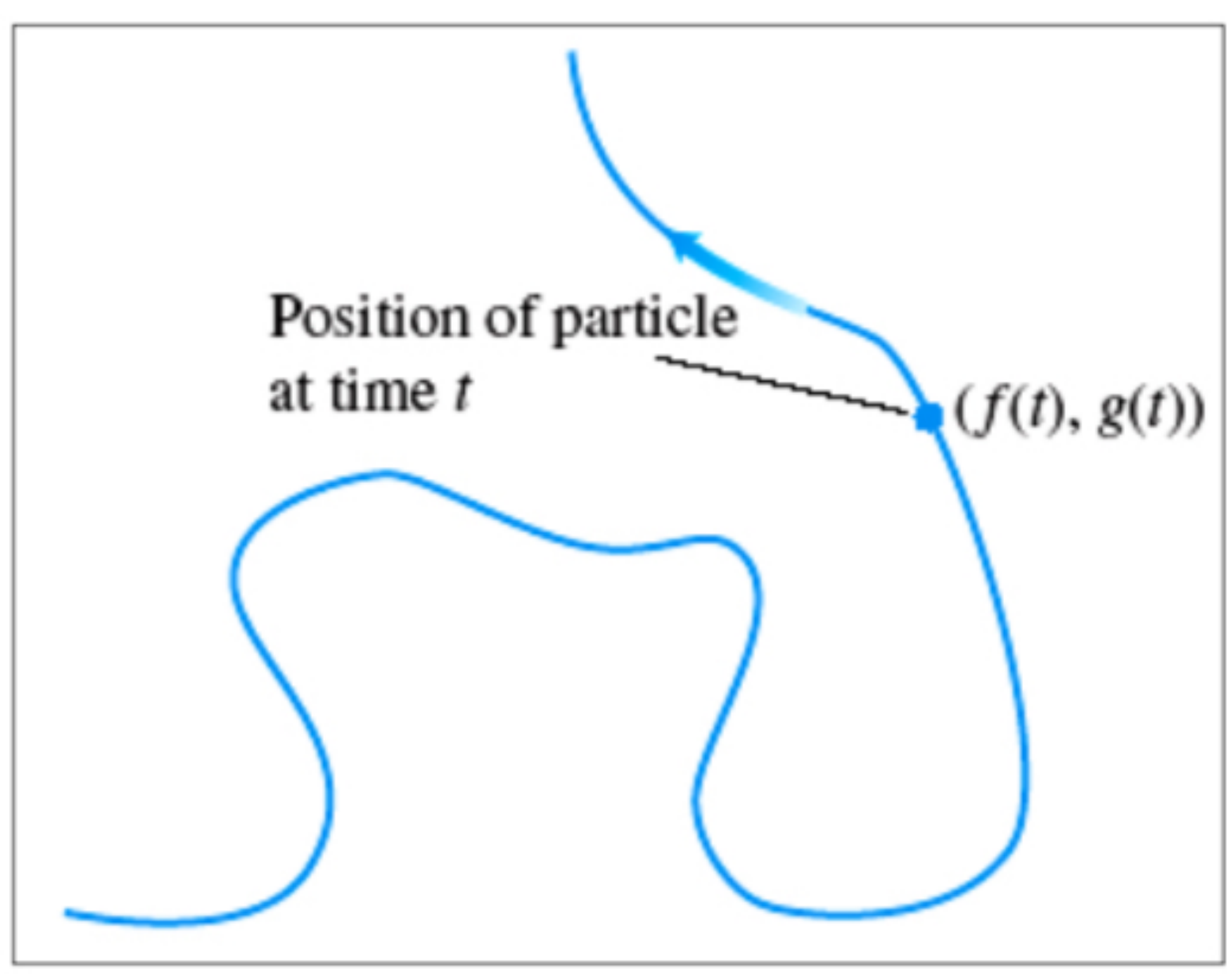

FIGURE 8 The path traced by a particle moving in the xy-plane is not always the graph of a function of x or a function of y.