3.6 Indeterminate Forms and L'Hôpital's Rule

John Bernoulli discovered a rule for calculating limits of fractions whose numerators and denominators both approach zero or $+\infty$. The rule is known today as **l'Hôpital's Rule**, after Guillaume de l'Hôpital. He was a French nobleman who wrote the first introductory differential calculus text, where the rule first appeared in print.

Indeterminate Form 0/0

If the continuous functions $f(x)$ and $g(x)$ are both zero at $x = a$, then

$$\lim_{x \to a} \frac{f(x)}{g(x)}$$

cannot be found by substituting $x = a$. The substitution produces 0/0, a meaningless expression, which we cannot evaluate. We use 0/0 as a notation for an expression known as an **indeterminate form**. Sometimes, but not always, limits that lead to indeterminate forms may be found by cancellation, rearrangement of terms, or other algebraic manipulations.

we calculate derivatives and which always produces the equivalent of 0/0 when we substitute $x = a$. L'Hôpital's Rule enables us to draw on our success with derivatives to evaluate limits that otherwise lead to indeterminate forms.

THEOREM L'Hôpital's Rule (First Form)

Suppose that $f(a) = g(a) = 0$, that $f'(a)$ and $g'(a)$ exist, and that $g'(a) \neq 0$. Then

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{f'(a)}{g'(a)}.$$

EXAMPLE 1 Using L'Hôpital's Rule

(a) $\displaystyle \lim_{x \to 0} \frac{3x - \sin x}{x} = \frac{3 - \cos x}{1}\bigg|_{x=0} = 2$

(b) $\displaystyle \lim_{x \to 0} \frac{\sqrt{1 + x} - 1}{x} = \frac{\frac{1}{2\sqrt{1 + x}}}{1}\Bigg|_{x=0} = \frac{1}{2}$ ■

Sometimes after differentiation, the new numerator and denominator both equal zero at $x = a$, as we see in Example 2. In these cases, we apply a stronger form of l'Hôpital's Rule.

THEOREM **L'Hôpital's Rule (Stronger Form)**

Suppose that $f(a) = g(a) = 0$, that f and g are differentiable on an open interval I containing a, and that $g'(x) \neq 0$ on I if $x \neq a$. Then

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)},$$

assuming that the limit on the right side exists.

EXAMPLE 2 Applying the Stronger Form of L'Hôpital's Rule

(a) $$\lim_{x\to 0} \frac{\sqrt{1+x} - 1 - x/2}{x^2} \qquad \frac{0}{0}$$

$$= \lim_{x\to 0} \frac{(1/2)(1+x)^{-1/2} - 1/2}{2x} \qquad \text{Still } \frac{0}{0}\text{; differentiate again.}$$

$$= \lim_{x\to 0} \frac{-(1/4)(1+x)^{-3/2}}{2} = -\frac{1}{8} \qquad \text{Not } \frac{0}{0}\text{; limit is found.}$$

(b) $$\lim_{x\to 0} \frac{x - \sin x}{x^3} \qquad \frac{0}{0}$$

$$= \lim_{x\to 0} \frac{1 - \cos x}{3x^2} \qquad \text{Still } \frac{0}{0}$$

$$= \lim_{x\to 0} \frac{\sin x}{6x} \qquad \text{Still } \frac{0}{0}$$

$$= \lim_{x\to 0} \frac{\cos x}{6} = \frac{1}{6} \qquad \text{Not } \frac{0}{0}\text{; limit is found.}$$

■

EXAMPLE 3 Incorrectly Applying the Stronger Form of L'Hôpital's Rule

$$\lim_{x\to 0} \frac{1 - \cos x}{x + x^2} \qquad \frac{0}{0}$$

$$= \lim_{x\to 0} \frac{\sin x}{1 + 2x} = \frac{0}{1} = 0 \qquad \text{Not } \frac{0}{0}\text{; limit is found.}$$

Up to now the calculation is correct, but if we continue to differentiate in an attempt to apply l'Hôpital's Rule once more, we get

$$\lim_{x\to 0}\frac{1-\cos x}{x+x^2}=\lim_{x\to 0}\frac{\sin x}{1+2x}=\lim_{x\to 0}\frac{\cos x}{2}=\frac{1}{2},$$

which is wrong. L'Hôpital's Rule can only be applied to limits which give indeterminate forms, and $0/1$ is not an indeterminate form. ■

EXAMPLE 4 Using L'Hôpital's Rule with One-Sided Limits

(a) $\displaystyle\lim_{x\to 0^+}\frac{\sin x}{x^2}$ $\quad \frac{0}{0}$

$\displaystyle=\lim_{x\to 0^+}\frac{\cos x}{2x}=\infty$ $\quad$ Positive for $x>0$.

(b) $\displaystyle\lim_{x\to 0^-}\frac{\sin x}{x^2}$ $\quad \frac{0}{0}$

$\displaystyle=\lim_{x\to 0^-}\frac{\cos x}{2x}=-\infty$ $\quad$ Negative for $x<0$. ■

Indeterminate Forms ∞/∞, $\infty\cdot 0$, $\infty-\infty$

Sometimes when we try to evaluate a limit as $x\to a$ by substituting $x=a$ we get an ambiguous expression like ∞/∞, $\infty\cdot 0$, or $\infty-\infty$, instead of $0/0$. We first consider the form ∞/∞.

In more advanced books it is proved that l'Hôpital's Rule applies to the indeterminate form ∞/∞ as well as to $0/0$. If $f(x)\to\pm\infty$ and $g(x)\to\pm\infty$ as $x\to a$, then

$$\lim_{x\to a}\frac{f(x)}{g(x)}=\lim_{x\to a}\frac{f'(x)}{g'(x)}$$

provided the limit on the right exists. In the notation $x\to a$, a may be either finite or infinite. Moreover $x\to a$ may be replaced by the one-sided limits $x\to a^+$ or $x\to a^-$.

EXAMPLE 5 Working with the Indeterminate Form ∞/∞

Find

(a) $\displaystyle\lim_{x\to\pi/2}\frac{\sec x}{1+\tan x}$

(b) $\displaystyle\lim_{x\to\infty}\frac{x-2x^2}{3x^2+5x}$

Solution

(a) The numerator and denominator are discontinuous at $x = \pi/2$, so we investigate the one-sided limits there. To apply l'Hôpital's Rule, we can choose I to be any open interval with $x = \pi/2$ as an endpoint.

$$\lim_{x\to(\pi/2)^-} \frac{\sec x}{1 + \tan x} \qquad \frac{\infty}{\infty} \text{ from the left}$$

$$= \lim_{x\to(\pi/2)^-} \frac{\sec x \tan x}{\sec^2 x} = \lim_{x\to(\pi/2)^-} \sin x = 1$$

The right-hand limit is 1 also, with $(-\infty)/(-\infty)$ as the indeterminate form. Therefore, the two-sided limit is equal to 1.

(b) $\displaystyle \lim_{x\to\infty} \frac{x - 2x^2}{3x^2 + 5x} = \lim_{x\to\infty} \frac{1 - 4x}{6x + 5} = \lim_{x\to\infty} \frac{-4}{6} = -\frac{2}{3}.$ ■

EXAMPLE 6 Working with the Indeterminate Form $\infty \cdot 0$

Find

$$\lim_{x\to\infty}\left(x \sin\frac{1}{x}\right)$$

Solution

$$\lim_{x\to\infty}\left(x \sin\frac{1}{x}\right) \qquad \infty \cdot 0$$

$$= \lim_{h\to 0^+}\left(\frac{1}{h}\sin h\right) \qquad \text{Let } h = 1/x.$$

$$= 1$$

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EXAMPLE 7 Working with the Indeterminate Form $\infty - \infty$

Find

$$\lim_{x\to 0}\left(\frac{1}{\sin x} - \frac{1}{x}\right).$$

Solution If $x \to 0^+$, then $\sin x \to 0^+$ and

$$\frac{1}{\sin x} - \frac{1}{x} \to \infty - \infty.$$

Similarly, if $x \to 0^-$, then $\sin x \to 0^-$ and

$$\frac{1}{\sin x} - \frac{1}{x} \to -\infty - (-\infty) = -\infty + \infty.$$

Neither form reveals what happens in the limit. To find out, we first combine the fractions:

$$\frac{1}{\sin x} - \frac{1}{x} = \frac{x - \sin x}{x \sin x} \qquad \text{Common denominator is } x \sin x$$

Then apply l'Hôpital's Rule to the result:

$$\begin{aligned}
\lim_{x \to 0}\left(\frac{1}{\sin x} - \frac{1}{x}\right) &= \lim_{x \to 0} \frac{x - \sin x}{x \sin x} && \frac{0}{0} \\
&= \lim_{x \to 0} \frac{1 - \cos x}{\sin x + x \cos x} && \text{Still } \frac{0}{0} \\
&= \lim_{x \to 0} \frac{\sin x}{2\cos x - x \sin x} = \frac{0}{2} = 0. && \blacksquare
\end{aligned}$$

EXERCISES 3.6

Finding Limits

In Exercises 1–6, use l'Hôpital's Rule to evaluate the limit. Then evaluate the limit using a method studied in Chapter 2.

1. $\lim_{x \to 2} \frac{x - 2}{x^2 - 4}$

2. $\lim_{x \to 0} \frac{\sin 5x}{x}$

3. $\lim_{x \to \infty} \frac{5x^2 - 3x}{7x^2 + 1}$

4. $\lim_{x \to 1} \frac{x^3 - 1}{4x^3 - x - 3}$

5. $\lim_{x \to 0} \frac{1 - \cos x}{x^2}$

6. $\lim_{x \to \infty} \frac{2x^2 + 3x}{x^3 + x + 1}$

Applying l'Hôpital's Rule

Use l'Hôpital's Rule to find the limits in Exercises 7–26.

7. $\lim_{t \to 0} \frac{\sin t^2}{t}$

8. $\lim_{x \to \pi/2} \frac{2x - \pi}{\cos x}$

9. $\lim_{\theta \to \pi} \frac{\sin \theta}{\pi - \theta}$

10. $\lim_{x \to \pi/2} \frac{1 - \sin x}{1 + \cos 2x}$

11. $\lim_{x \to \pi/4} \frac{\sin x - \cos x}{x - \pi/4}$

12. $\lim_{x \to \pi/3} \frac{\cos x - 0.5}{x - \pi/3}$

13. $\lim_{x \to (\pi/2)^-} \left(x - \frac{\pi}{2}\right) \tan x$

14. $\lim_{x \to 0} \frac{2x}{x + 7\sqrt{x}}$

15. $\lim_{x \to 1} \frac{2x^2 - (3x + 1)\sqrt{x} + 2}{x - 1}$

16. $\lim_{x \to 2} \frac{\sqrt{x^2 + 5} - 3}{x^2 - 4}$

17. $\lim_{x \to 0} \frac{\sqrt{a(a + x)} - a}{x}, \quad a > 0$

18. $\lim_{t \to 0} \frac{10(\sin t - t)}{t^3}$

19. $\lim_{x \to 0} \frac{x(\cos x - 1)}{\sin x - x}$

20. $\lim_{h \to 0} \frac{\sin(a + h) - \sin a}{h}$

21. $\lim_{r \to 1} \frac{a(r^n - 1)}{r - 1}$, n a positive integer

22. $\lim_{x \to 0^+} \left(\frac{1}{x} - \frac{1}{\sqrt{x}}\right)$

23. $\lim_{x \to \infty} (x - \sqrt{x^2 + x})$

24. $\lim_{x \to \infty} x \tan \frac{1}{x}$

25. $\lim_{x \to \pm\infty} \frac{3x - 5}{2x^2 - x + 2}$

26. $\lim_{x \to 0} \frac{\sin 7x}{\tan 11x}$