* Unbias *

Mathematical Expectation:-

let X be a r.v. with Probability function $f(x)$ and $u(x)$ be a real-valued function of X. Then the expected value of $u(x)$ is given by:-

$$E[u(x)] = \begin{cases} \sum_{x=-\infty}^{\infty} u(x).f(x) & , \text{ discrete} \\ \int_{x=-\infty}^{\infty} u(x).f(x) & , \text{ Continuous} \end{cases}$$

Notes:-

1- if c is a constant, then $E(c) = c$

2- if c is a constant and u is a function of X then $E[c\,u(x)] = c\,E[u(x)]$

3- if $c_1, \dots, c_n$ are constants and $u_1, u_2, \dots, u_n$ are functions, then $E(\sum_{i=1}^{n} c_i u_i) = \sum_{i=1}^{n} E(c_i u_i)$

$$= \sum_{i=1}^{n} c_i\,E(u_i)$$

Ex) let $X_i \sim U(0,\theta)$, $i=1,2,3$, is $\hat{\theta} = Y_3$ an unbiased for θ? where Y_3 is O.S. Find the Unbiased estimator for θ?

Solution) $E(\hat{\theta}) = E(Y_3)$ → Y_3 is O.S.

must find $g(y_3) = ?$

$\because X \sim U(0,\theta) \rightarrow f(x) = \frac{1}{\theta}$, $0 < x < \theta$

$g(y_n) = n\,(F_x(y_n))^{n-1} f_x(y_n)$ ← Max قانون

Ex) $n=3$ $\quad Y_3$ is o.s.

$$f(x)=\frac{1}{\theta} \rightarrow f_x(y_3)=\frac{1}{\theta}$$

$$F(x)=\frac{1}{\theta}\int_0^x f(u)\,du=\frac{x}{\theta} \rightarrow F_x(y_3)=\frac{y_3}{\theta}$$

Then $g(y_3)=3\left[\frac{y_3}{\theta}\right]^2\cdot\frac{1}{\theta}=\begin{cases}\frac{3y_3^2}{\theta^3}, & 0<y_3<\theta \\ 0, & \text{o.w.}\end{cases}$

$$\rightarrow E(y_3)=\int_0^\theta y_3\, g(y_3)\,dy_3=\int_0^\theta y_3\cdot\frac{3y_3^2}{\theta^3}\,dy_3 \Rightarrow \frac{3}{\theta^3}\int_0^\theta y_3\, y_3^2\,dy_3$$

$$\frac{3}{\theta^3}\int_0^\theta y_3^3\,dy_3 = \frac{y_3^4}{4}\Big]_0^\theta \quad \frac{3}{\theta^3}\left[\frac{\theta^4}{4}-0\right]=\frac{3\theta}{4}$$

$$\rightarrow E(y_3)=\frac{3}{4}\theta \qquad \therefore E(\hat{\theta})\neq\theta$$

$\therefore \hat{\theta}=y_3$ is ~~un~~biased estimator for θ.

Note:- The unbiased estimator will be $\hat{\theta}=\frac{4}{3}y_3$.

H.W) let $x_1, x_2, \dots, x_n$ be a r.s. with $E[x_i]=\mu$ and $\sigma(x_i)=\sigma^2$, Show that

$$S^2=\frac{1}{n-1}\sum_{i=1}^{n}(x_i-\bar{x})^2 \text{ is unbiased for } \sigma^2?$$

Note:- $E(\bar{x})=\mu$, $\bar{x}$ is unbiased for μ.

$E(s^2)=\sigma^2$, s^2 is unbiased for σ^2.