[1]

الكفاية * Sufficient Statistics *

likelihood Function :-

The likelihood Function of a r.s. $X_1, \ldots, X_n$ of size (n) From a distribution with pdf $f(x)$ with parameters $\theta_1, \theta_2, \ldots, \theta_n$ is defined to be the joint pdf of the n. r.v's $X_1, X_2, \ldots, X_n$ which considered as a function of θ's and denoted by $L(\bar{\theta}, \bar{x})$:-

$$L(\hat{\theta}, \bar{x}) = L(\theta_1, \theta_2, \ldots, \theta_n; x_1, x_2, \ldots, x_n)$$

$$= f(\bar{x}, \bar{\theta}) = \prod_{i=1}^{n} f(x_i)$$

التوزيع الاحتمالي الشرطي

$$\pi(x_1, \ldots, x_n / t) = \frac{L(\theta; x_1, x_2, \ldots, x_n)}{g(t; \theta)}$$

Conditional Distribution

if T is a S.S. for θ, then

$$= \pi(x_1, \ldots, x_n; t)$$

* Conditional Distribution *

let X and y be any Two r.v. with joint density function $f(x,y)$, The conditional probability density Function of x given $Y = y$ is defined by :-

$$f(x \mid Y = y) = \frac{f(x,y)}{f_2(y)} \quad , f_2(y) > 0$$

∴ Sufficient Statistics ∴

Sufficient Statistics can solved by Two ways :-

By Definition

By Using Neyman's Factorization theorem.

1) By Using Definition:-

احصاء تحقق على العينة

let $X_1, X_2, \dots, X_n$ be a r.s. From dis. $F(x,\theta)$ and let $T = u(X_1, \dots, X_n)$ be a statistics whose pdf is $g(t,\theta)$:-

T الاحصائية pdf

* A sufficient Statistics, T is a statistics which contains all the information for the estimation of θ.

* A sufficient Statistics in not unique.

* The conditional distribution of sample r.v's given the value of T of T is defined as:- اذا كانت النتيجة بعد تطبيقنا القانون خالي من الاختصار

نقول عن الناتج احصاء جديد لا تحقق على المعلمة فقط في هذه الحالة يكون sufficient

$$\pi(x_1, \dots, x_n / t) = \frac{F(x_1, \dots x_n, t; \theta)}{g(t,\theta)} = H(x_1, \dots, x_n)$$

دالة مشتركة

$$= \frac{L(\theta; x_1, \dots x_n)}{g(t,\theta)}$$

Notes :-

c is Constant → 1) $\prod_{i=1}^{n} c = (c)^n$.

2) $\prod_{i=1}^{n} e^{x} = e^{\sum_{i=1}^{n} x_i}$.

3) $\prod_{i=1}^{n} a^{x} = a^{\sum_{i=1}^{n} x_i}$.

4) $\prod_{i=1}^{n} e^{a} = e^{na}$.

5) $\sum_{i=1}^{n} m = mn$.

Ex 1) let $X_i, i=1,\dots,n$ be a r.s. From Poisson dist. with Parameter θ. Show that $T=\sum_{i=1}^{n} x_i$ is S.S. for θ?

Solution) By Using Definition.

product $\rightarrow \pi(x_1,\dots,x_n/t) = \dfrac{L(\theta;x_1,\dots,x_n)}{g(t;\theta)}$

$\because X_i \sim P(\theta) \rightarrow f(x,\theta)=\begin{cases} \dfrac{e^{-\theta}\cdot\theta^{x}}{x!} & , x=0,1,\dots \\ 0 & , o.w. \end{cases}$

where $L(\theta;x)=f(x_1,\theta).f(x_2,\theta)\dots f(x_n,\theta)$

$$=\frac{e^{-\theta}\cdot\theta^{x_1}}{x_1!}\cdot\frac{e^{-\theta}\cdot\theta^{x_2}}{x_2!}\cdots\frac{e^{-\theta}\cdot\theta^{x_n}}{x_n!}$$

$$=\prod_{i=1}^{n}\left[\frac{e^{-\theta}\cdot\theta^{x_i}}{x_i!}\right]=\frac{e^{-\theta n}\cdot\theta^{\sum_{i=1}^{n}x_i}}{(\prod_{i=1}^{n}x_i)!} \leftarrow$$ نتيجة الربط فقط

Since $X\sim P(\theta) \rightarrow T=\sum_{i=1}^{n}X_i \sim P(n\theta)$ الكراسة (7)

From Properties $\nearrow$
(الفصل الاول)

$$\Rightarrow g(t;\theta)=\frac{e^{-n\theta}\cdot(n\theta)^{t}}{t!} \Leftarrow$$ بالاعتماد على الخطوة السابقة نا علي الحالة θ ونعوضها $n\theta$

$$\Rightarrow \pi(x_1,x_2,\dots,x_n/t)=\frac{\dfrac{e^{-n\theta}\cdot\theta^{\sum_{i=1}^{n}x_i}}{(\prod_{i=1}^{n}x_i)!}}{\dfrac{e^{-n\theta}\cdot(n\theta)^{t}}{t!}}=\frac{\dfrac{\cancel{e^{-n\theta}}\ \cancel{\theta^{t}}}{(\prod_{i=1}^{n}x_i)!}}{\dfrac{\cancel{e^{-n\theta}}\cdot(n\cancel{\theta})^{t}}{t!}}$$

$$=\frac{\dfrac{1}{(\prod_{i=1}^{n}x_i)!}}{\dfrac{(n)^{t}}{t!}}$$

$\therefore$ T is a S.S. for θ.

Ex2) let $X_1, X_2, \dots, X_n$ be a r.s. from $Ber(\theta)$, show that
$T = \sum_{i=1}^{n} x_i$ is a S.S. for θ?

Solution) $\because x_i \sim Ber(\theta)$

$$\rightarrow f(x;\theta) = \begin{cases} \theta^{x}(1-\theta)^{1-x} & , x = 0,1,\dots,n \\ 0 & , o.w \end{cases}$$

$$\prod f(x_i;\theta) = f(x_1,\theta)\dots f(x_n,\theta)$$
$$= \theta^{x_1}(1-\theta)^{1-x_1} \dots \theta^{x_n}(1-\theta)^{1-x_n}$$
$$= \prod_{i=1}^{n} \left(\theta^{x_i}(1-\theta)^{1-x_i}\right)$$
$$= \theta^{\sum_{i=1}^{n} x_i} \cdot (1-\theta)^{n-\sum_{i=1}^{n} x_i} = \theta^{t} \cdot (1-\theta)^{t-n}$$

خاصية (م ٢) الفصل الاول $\because x_i \sim Ber(\theta) \rightarrow Y = \sum_{i=1}^{n} x_i \sim Bin(n,\theta)$

$$\rightarrow T = \sum_{i=1}^{n} x_i \sim Bin(n,\theta)$$

$$\therefore g(t;\theta) = C^{n}_{t}\, \theta^{t}(1-\theta)^{n-t}$$

قانون S.S. $\Rightarrow \pi(x_1,\dots,x_n / t) = \dfrac{\theta^{t} \cdot (1-\theta)^{n-t}}{C^{n}_{t}\, \theta^{t}(1-\theta)^{n-t}} = \dfrac{1}{C^{n}_{t}}$

$\therefore T = \sum_{i=1}^{n} x_i$ is S.S. for θ.