[5]

* Second way : Neyman's Factorization theorem *

let $X_1, \dots X_n$ be a r.s. from a distribution has pdf $f(x;\theta)$ and $T = t(X_1, \dots, X_n)$ be a statistics whose pdf $g(t;\theta)$

تقسم الى جزئين

$$\rightarrow L(\theta) = g(t;\theta) \cdot h(x_1, \dots, x_n).$$

جزء يحتوي على المعلمة
والجزء الاخر لا يحتوي على المعلمة.

where g and h are non-negative functions.

Ex1) let $X_i \sim N(\theta, \sigma^2)$, $i = 1, 2, \dots, n$, Show that

$T = \sum_{i=1}^{n} x_i$ is S.S. for θ?

Solution:- By using Neyman's Factorization theorem.

$$L(\theta; X) = f(x_1;\theta) \cdot f(x_2,\theta) \dots f(x_n,\theta)$$

$$\because X_i \sim N(\theta, \sigma^2) \rightarrow f(x;\theta) = \begin{cases} \frac{1}{\sigma \cdot \sqrt{2\pi}} \cdot e^{-\frac{1}{2}\left(\frac{x-\theta}{\sigma^2}\right)^2}, & x \in \mathbb{R} \\ 0, & \text{o.w.} \end{cases}$$

$$\therefore L(X;\theta) = f(x_1,\theta) \dots f(x_n,\theta) = \prod_{i=1}^{n} f(x_i;\theta)$$

$$= \prod_{i=1}^{n} \left(\frac{1}{\sigma \cdot \sqrt{2\pi}} \cdot e^{-\frac{1}{2}\frac{(x-\theta)^2}{\sigma^2}} \right)$$

$$= \left(\frac{1}{\sigma \cdot \sqrt{2\pi}}\right)^n \cdot e^{-\frac{1}{2\sigma^2} \cdot \sum_{i=1}^{n} (x_i - \theta)^2}$$

$$\sum_{i=1}^{n} (x_i^2 - 2\theta x_i + \theta^2)$$

$$\left(\sum_{i=1}^{n} x_i^2 - \underbrace{2\theta \sum_{i=1}^{n} x_i + n\theta^2}\right)$$

$$L(x;\theta) = g(t;\theta) \cdot h(x_1, \dots, x_n)$$

يحتوي على $\hat{\theta}$

$$\rightarrow g(t;\theta) = e^{-\frac{1}{2\sigma^2} \cdot \left(-2\theta \sum_{i=1}^{n} x_i + n\theta^2\right)} \leftarrow g(x,t) \text{ هو}$$

$e^{-\frac{1}{2\sigma^2}(\sum x_i^2)}$ فالمتبقي هو h هو

$$h(x_1, \dots, x_n) = e^{-\frac{1}{2\sigma} \cdot \left(\sum_{i=1}^{n} x_i\right)}$$

$$\therefore T = \sum_{i=1}^{n} x_i \text{ is S.S. for } \theta.$$

Ex2) let $x_1, x_2, \dots, x_n$ be a r.s. from Poisson dist. with parameter θ, show that $T = \sum_{i=1}^{n} x_i$ is S.S. for θ? [by using Neyman's th.]

Solution) By using Neyman's Factorization theorem.

$$\because x_i \sim P(\theta) \Rightarrow f(x;\theta) = \begin{cases} \dfrac{e^{-\theta} \cdot \theta^{x}}{x!} \\ 0, \quad \text{o.w.} \end{cases}$$

$$L(x;\theta) = g(t;\theta) \cdot h(x_1, \dots, x_n)$$

$$L(x;\theta) = \prod_{i=1}^{n} f(x_i, \theta) \leftarrow$$

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$$\prod_{i=1}^{n} f(x_i, \theta) = \frac{e^{-n\theta} \cdot \theta^{\sum_{i=1}^{n} x_i}}{\left(\prod_{i=1}^{n} x_i\right)!}$$

$$L(x;\theta) \Rightarrow g(t;\theta) = e^{-n\theta} \cdot \theta^{\sum_{i=1}^{n} x_i}$$

$$h(x_1, \dots, x_n) = \frac{1}{\left(\prod_{i=1}^{n} x_i\right)!}$$

$$\therefore T = \sum_{i=1}^{n} x_i \text{ is S.S. for } \theta.$$

[7]

H.W) let $f(x,\theta) = \theta . x^{\theta-1}$, $0 < x < 1$

is $T = \sum_{i=1}^{n} x_i$ is S.S. for θ?