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الفصل الثالث

4. Completeness التمام

Let $\{f(x,\theta),\ \theta \in \Omega\}$ be a family of pdf and $U(x)$ be an arbitrary function of X not depending on θ. if
دالة عشوائية

$$E_\theta[U(X)] = 0 \text{ for all } \theta \in \Omega$$

requires That The function itself equal to (0) for all possible values of X; Then we say That This family is a complete family of pdfs

$$E_\theta[U(X)] = 0 \text{ for all } \theta \in \Omega \Rightarrow U(x) = 0 \text{ for all } x$$

- Exponential Family of densities

- Theorem: Let $X_i;\ i=1,\dots,n$ be a random sample from a population with pdf $f(x,\theta),\ \theta \in \Omega$, $f(x,\theta)$ is called a member of exp. family if it can be written in The following form

$$f(x,\theta) = \exp[\ln f(x,\theta)]$$
$$= \exp[a(\theta) + b(x) + c(\theta)\, d(x)]$$

That $f(x,\theta)$ is one parameter exponential family Then $T = \sum_{i=1}^{n} d(x_i)$ is a complete sufficient statistic for θ.

- Note

① $\ln(ab) = \ln a + \ln b$ ④ $\ln e^x = x$

② $\ln\left(\frac{a}{b}\right) = \ln a - \ln b$ ⑤ $e^{\ln x} = x$

③ $\ln(a^b) = b \ln a$

ملاحظة // تكون دالة الـ exp في حالتين
اذا كان هناك معلمة واحدة وايضاً في حال اذا كان هناك معلمتين كالامثلة الاتية.

EX 1/ let $X_i \sim N(\theta, \sigma^2)$, $i = 1, 2, \dots, n$ Show That $f(x, \theta, \sigma^2)$ belong to The exp. family and Find $T = t(x)$ The Complete suff. stat. for θ.

Sol/

if we can write $F(x, \theta)$ as

$$F(x, \theta) = \exp[\ln F(x, \theta)]$$

$$= \exp\left[\ln\left(\frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\frac{(x-\theta)^2}{\sigma^2}}\right)\right]$$

$$= \exp\left[\ln\left(\frac{1}{\sigma\sqrt{2\pi}}\right) \cdot \ln\left(e^{-\frac{1}{2}\frac{(x-\theta)^2}{\sigma^2}}\right)\right]$$

مربع حدانية

$$= \exp\left[\ln\left(\frac{1}{\sigma\sqrt{2\pi}}\right) - \frac{1}{2\sigma^2}(x-\theta)^2\right]$$

$$= \exp\left[\ln\left(\frac{1}{\sigma\sqrt{2\pi}}\right) - \frac{x^2}{2\sigma^2} + \frac{1}{2\sigma^2} 2\theta x - \frac{1}{2\sigma^2}\theta^2\right)$$

where $a(\theta) = \ln\left(\frac{1}{\sigma\sqrt{2\pi}}\right) - \frac{1}{2\sigma^2}\theta^2$

$b(x) = \frac{x^2}{2\sigma^2}$, $c(\theta) = \frac{1}{2\sigma^2} 2\theta$ and

$d(x) = x$

Then $F(x, \theta, \sigma^2)$ belong to exp. family

and $T = \sum_{i=1}^{n} d(x_i) = \sum_{i=1}^{n} x_i$

is a Complete Sufficient statistic for θ.

Ex2 let $X_i \sim exp(\theta)$, $i=1,2,\dots,n$ ① Show That $F(x,\theta)$ belong to The exp. family ② Find The Comp. Suff. Stat. for θ.

Sol/

① $F(x,\theta) = exp(\ln F(x,\theta))$

$$= exp\left[\ln\left(\frac{1}{\theta} e^{-\frac{x}{\theta}}\right)\right]$$

$$= exp\left[\ln\left(\frac{1}{\theta}\right) - \frac{x}{\theta}\right]$$

$a(\theta) = \ln\left(\frac{1}{\theta}\right)$, $b(x) = 0$, $c(\theta) = \frac{-1}{\theta}$, $d(x) = x$

Then $f(x,\theta)$ belong to The exp. family.

② $T = \sum_{i=1}^{n} d(x_i) = \sum_{i=1}^{n} x_i$ is Comp. Suff. Stat. for θ

Ex3 let $X_i \sim Ber(\theta)$ $i=1,2,\dots,n$ Find T The Comp. Suff. Stat. for θ

Sol/

$F(x,\theta) = exp[\ln F(x,\theta)]$

$$= exp\left[\ln\left(\theta^{x}(1-\theta)^{1-x}\right)\right]$$

$$= exp\left[x\ln(\theta) + (1-x)\ln(1-\theta)\right]$$

$$= exp\left[x\ln(\theta) + \ln(1-\theta) - x\ln(1-\theta)\right]$$

$$= exp\left[x(\ln(\theta) - \ln(1-\theta)) + \ln(1-\theta)\right]$$

$a(\theta) = \ln(1-\theta)$, $b(x) = 0$, $c(\theta) = \ln(\theta) - \ln(1-\theta)$

$d(x) = x$

The $F(x,\theta)$ belong to The exp. family and $T = \sum d(x_i)$

$= \sum x_i$ is Comp. S.S. for θ.