# 5- Rao - Blackwell Theorem.

Let $\hat{\theta}$ be an unbiased estimator and $T$ is The sufficient statistic of $\theta$ iff.

$$\tilde{\theta} = E(\hat{\theta}/T)$$

Then

1. $\tilde{\theta}$ is a function of suff. statistic.
2. $E(\tilde{\theta}) = \theta$ (unbiased).
3. $V(\tilde{\theta}) \leq V(\hat{\theta})$ (minimum Variance).

Ex1/ let $X_i \sim P(\theta)$; $i=1,\dots,n$ if $\hat{\theta} = X_1$ is unbiased estimator for $\theta$. Can we find a better estimator for $\theta$ by R-B Theorem?

Sol/

1- $\hat{\theta} = X_1$ is unbiased est. for $\theta$ where since $X_1 \sim P(\theta)$
Then $E(\hat{\theta}) = E(X_1) = \theta$

2- By Factorization Theorem

$$f(x_1,\dots,x_n,\theta) = \frac{e^{-n\theta}\,\theta^{\sum x_i}}{(\prod_{i=1}^{n} x_i)!} \quad \text{where}$$

$$g(t,\theta) = e^{-n\theta}\,\theta^{\sum x_i} \quad \text{and} \quad h(x_1,\dots,x_n) = \frac{1}{(\prod_{i=1}^{n}(x_i))!}$$

That means $T = \sum_{i=1}^{n} x_i$ is s.s. for $\theta$

3- By Rao-Blackwell Theorem $\tilde{\theta} = E(\hat{\theta}/T)$ is better estimator with minimum variance and unbiased estimator for $\theta$ such That.

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$$\tilde{\theta} = E(\hat{\theta}/T) = E\left(X_1 / \sum_{i=1}^{n} x_i = t\right) = \cancel{\Sigma x\ldots} \leftarrow$$ لان التوزيع متقطع فالتوقع قانونه هو.

قانون التوقع الشرطي $E(X/Y)$

$$= \Sigma\, x . P(x|y) \text{ متقطع}$$

$$\text{مستمر} \rightarrow \int_{-\infty}^{\infty} x . f(x|y)$$

Now

$$= \sum_{x_1} x_1\, f\left(X_1 = x_1 / \sum_{i=1}^{n} x_i = t\right) \qquad \Sigma\, x . f(x,y)$$

دالة شرطية

$$f\left(X_1 = x_1 / \sum_{i=1}^{n} x_i = t\right) = \frac{P\left(X_1 = x_1 , \sum_{i=1}^{n} x_i = t\right)}{f\left(\sum_{i=1}^{n} x_i = t\right)} \leftarrow \frac{\text{Joint}}{\text{Marginal}}$$

$$= \frac{P\left(X_1 = x_1 , \sum_{2}^{n} x_i = t - x_1\right)}{f\left(\sum_{1}^{n} x_i = t\right)} = \frac{P(X_1 = x_1) . P\left(\sum_{2}^{n} x_i = t - x_1\right)}{f\left(\sum_{i=1}^{n} x_i = t\right)}$$

where

$$f(X_1 = x_1) = \frac{e^{-\theta} . \theta^{x_1}}{(x_1)!}$$

$$\sum_{1}^{n} x_i = t \sim P(n\theta) \rightarrow P\left(\sum_{1}^{n} x_i = t\right) = \frac{e^{-n\theta} (n\theta)^t}{t!} \leftarrow$$ من الخصائص في الفصل ①

$$\sum_{2}^{n} x_i = t - x_1 \sim P((n-1), \theta)$$

قانون التوزيع هو

$$f\left(\sum_{2}^{n} x_i = t - x_1\right) = \frac{e^{-(n-1)\theta} ((n-1)\theta)^{t - x_1}}{(t - x_1)!} \qquad C_x^n = \frac{n!}{x!(n-x)!}$$

$$\therefore P\left[X_1 = x_1 / \sum_{1}^{n} x_i = t\right]$$

$$= \frac{\theta^{x_1} e^{-\theta}}{x_1!} . \frac{e^{-(n-1)\theta} ((n-1),\theta))^{t - x_1}}{(t - x_1)!} . \frac{t!}{e^{-n\theta} (n\theta)^t}$$

$$= \frac{(n-1)^{t - x_1}}{n^t} . \frac{t!}{x_1! (t - x_1)!} = C_{x_1}^{t} \frac{(n-1)^{t - x_1}}{n^{t - x_1 + x_1}} \leftarrow$$ بأضافة وطرح $x_1$

$$= \underbrace{C_{x_1}^{t} \left(\frac{n-1}{n}\right)^{t - x_1} \left(\frac{1}{n}\right)^{x_1}}_{\text{P.d.f binomial.}} \sim b\left(t, \frac{1}{n}\right)$$

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$\therefore E[X_1 / \sum_1^n x_i = t] = \sum x_i \, f(x_1 / \sum x_i = t)$

$= \sum x_1 \, C_{x_1}^{t} \left(\frac{1}{n}\right)^{x_1} \left(1 - \frac{1}{n}\right)^{t - x_1}$

bin توقع بي توزيع

$= t \cdot \frac{1}{n} = \frac{\sum x_i}{n} = \bar{X}$ .

$\therefore \tilde{\theta} = \bar{X}$

① $\tilde{\theta}$ is a function of S.S

② $E(\tilde{\theta}) = E(\bar{X}) = \theta$ (unbiased)

③ $V(\tilde{\theta}) = V(\bar{X}) = \frac{n\theta}{n^2} = \frac{\theta}{n}$

$E(\bar{X}) = E\left(\frac{\sum x_i}{n}\right)$

$= \frac{1}{n} \sum E(x_i) \leftarrow$ توزيع $P(\theta)$

$= \frac{1}{\cancel{n}} \cdot \cancel{n} \theta$

$V(\hat{\theta}) = V(x_1) = \theta$

$\therefore V(\tilde{\theta}) \leq V(\hat{\theta})$ (minimum Variance)