6- Efficiency الكفاءة، الفعالية

Def:- If $V(\hat{\theta})$ is equal to CRLB Then $\hat{\theta}$ is Called efficient estimator for θ. i.e

$$eff = \frac{CRLB}{V(\hat{\theta})}$$

- Cramer-Rao Lower Bound

$$CRLB = \frac{1}{I} = I^{-1}$$

where I is called by Fisher The amount of information about θ Contained in observation X.

$$I = nE\left[\frac{\partial[\ln(f(x,\theta)]}{\partial\theta}\right]^2 \quad \text{OR}$$

$$I = -nE\left[\frac{\partial^2(\ln(f(x,\theta))}{\partial\theta^2}\right] \quad \text{and} \quad V(\hat{\theta}) \geq I^{-1}$$

Ex.

let $X_i \sim p(\theta), i=1,2,\ldots,n$ Find CRLB, and show That $\hat{\theta} = \bar{x}$ is efficient estimator for θ.

Sol/

Since The range of X dose not depend on The unknown parameter θ which we wish to estimate, we Can proceed to Compute and use The Cramer-Rao Lower Bound for unbaised estimators:

$$CRLB = \frac{1}{I} = \frac{1}{nE\left[\frac{\partial(\ln(f(x,\theta)))}{\partial\theta}\right]^2}$$

Since $X \sim p(\theta)$

$$f(x,\theta) = \frac{e^{-\theta}\cdot\theta^{x}}{x!} \Rightarrow \ln(f(x,\theta)) = \ln\left(\frac{e^{-\theta}\cdot\theta^{x}}{x!}\right)$$

$$= -\theta + x\ln\theta - \ln(x!)$$

x يعتبر ثابت ومشتقة $\ln\theta$ هي $\frac{1}{\theta}$ ← x

$$\frac{\partial(\ln f(x,\theta))}{\partial\theta} = -1 + \frac{x}{\theta} = \frac{x-\theta}{\theta}$$

$$nE\left[\frac{\partial(\ln(f(x,\theta)))}{\partial\theta}\right]^2 = nE\left[\frac{x-\theta}{\theta}\right]^2 \Rightarrow \frac{n}{\theta^2}E(x-\theta)^2$$

θ ثابت

من التوزيع $p(\theta)$ نلاحظ أن:

$E(x) = \theta$ نستبدل θ بتوقع x

$\frac{n}{\theta^2}E[X-E(x)]^2$ $V(X) = \theta$

↑ هنا هو تباين $V(x)$

$$= \frac{n}{\theta^2}V(x) = \frac{n}{\theta^2}\cdot\theta = \frac{n}{\theta}$$

$$CRLB = \frac{1}{I} = \frac{1}{\frac{n}{\theta}} = \frac{\theta}{n}$$

- For efficient we have That $eff = \frac{CRLB}{V(\hat{\theta})}$

$$V(\hat{\theta}) = V(\bar{X}) = \frac{\sum_{i=1}^{n} V(X_i)}{n^2} = \frac{\sum_{i=1}^{n}\theta}{n^2} = \frac{n\theta}{n^2} = \frac{\theta}{n}$$

$\therefore eff = \frac{\theta/n}{\theta/n} = 1$ Then $\hat{\theta} = \bar{X}$ is efficient estimator for θ.

واجب (HW)

let $X_i \sim Exp(\theta)$, $i = 1, 2, \dots, n$ Find CRLB and show That $\hat{\theta} = \bar{X}$ is efficient estimator for θ.