7. Consistency الاتساق

Def: An estimator $\hat{\theta}$ is called a consistent estimator for θ iff.

1- $E(\hat{\theta}) = \theta$ i.e $\hat{\theta}$ is unbiased estimator for θ.

2- $\lim\limits_{n \to \infty} Var(\hat{\theta}) = 0$

Ex1/ let $X_i \sim N(\theta, 1)$, $i = 1, 2, \dots, n$ is $\hat{\theta} = \bar{x}$ a consistent estimator for θ? Note That $\bar{x} \sim N(\theta, \frac{1}{n})$

Sol/

① $E(\hat{\theta}) = E(\bar{x}) = \frac{1}{n} \sum E(x_i) = \frac{n\theta}{n} = \theta$

So $\hat{\theta}$ is unbiased estimator for θ.

$\bar{x} \sim N(\mu, \sigma^2) \Rightarrow V(x) = \sigma^2$

② $\lim\limits_{n \to \infty} Var(\hat{\theta}) = \lim\limits_{n \to \infty} Var(\bar{x}) = \lim\limits_{n \to \infty} \frac{1}{n} = 0$ $\quad \bar{x} \sim (\theta, \frac{1}{n})$

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$\therefore \hat{\theta} = \bar{x}$ is a consistent estimator for θ.

Ex2 let $x_i \sim Bin(1, \theta)$, $i = 1, \dots, n$ is $\hat{\theta} = \bar{x}$ a consistent estimator for θ?

Sol/

① $E(\hat{\theta}) = E(\bar{x}) = \frac{1}{n} \sum E(x_i) = \frac{n\theta}{n} = \theta$

$\hat{\theta} = \bar{x}$ is unbiased estimator for θ.

② $\lim\limits_{n \to \infty} var(\hat{\theta}) = \lim\limits_{n \to \infty} var(\bar{x}) \Rightarrow \lim\limits_{n \to \infty} var\left(\frac{\sum x_i}{n}\right)$

$\lim\limits_{n \to \infty} \frac{\sum V(x_i)}{n^2} = \lim\limits_{n \to \infty} \frac{\sum \theta(1-\theta)}{n^2} = \lim\limits_{n \to \infty} \frac{\cancel{n}\, \theta(1-\theta)}{n^{\cancel{2}}}$

$= \lim\limits_{n \to \infty} \frac{\theta(1-\theta)}{n} = 0$ $\therefore \hat{\theta}$ is a consistent estimator for θ.

Ex3 let $x_i \sim Exp(\theta)$ $i=1,2,\dots,n$ is $\hat{\theta}=\bar{x}$ a consistent estimator for θ?

Sol/

① $E(\hat{\theta}) = E(\bar{x}) = \dfrac{\sum E(x_i)}{n} = \dfrac{n\theta}{n} = \theta$

$\hat{\theta}$ is unbiased estimator for θ.

② $\lim\limits_{n\to\infty} Var(\hat{\theta}) = \lim\limits_{n\to\infty} Var(\bar{x}) = \lim\limits_{n\to\infty} \dfrac{\sum V(x_i)}{n^2}$

$\lim\limits_{n\to\infty} \dfrac{\theta^2}{n} = 0$ $\therefore$ $\hat{\theta}$ is a consistent estimator for θ.

H.W let $x_i \sim P(\theta)$ $i=1,2,\dots,n$ is $\hat{\theta}=\bar{x}$ a consistent estimator for θ?