طريقة تقدير الإمكان الأعظم

② Maximum likelihood Estimation Method.

Suppose That X is a r.v. with probability distiribution $f(x,\theta)$ where θ is a single unknown parameter. Let $x_1, x_2, \dots, x_n$ be The observed values in a random Sample of size n, Then The likelihood function of The Sample is

$$L(\theta) = f(x_1,\theta) \cdot f(x_2,\theta) \cdot \dots \cdot f(x_n,\theta) = \prod_{i=1}^{n} f(x_i,\theta)$$

Note That The likelihood function is now a function of only The unknown parameter θ.

The maximum likelihood estimator (MLE) of θ is The value of θ That maximizes The likelihood function $L(\theta)$

The method steps.

خطوات الحل

1- Find the likelihood function.

1) إيجاد $\pi(L(\theta))$

$$L(\theta) = f(x_1, \theta) \cdot f(x_2, \theta) \cdots f(x_n, \theta)$$

2) إيجاد $\ln \pi(L(\theta))$

3) الاشتقاق بالنسبة للمعلمة.

2- maximize the log of the likelihood function

4) مساواة المقدار بالخطوة (3) الى الصفر وفي هذه الحالة يتم تحويل اي معلمة الى مقدار اي اذا كانت θ تحول الى $\hat{\theta}$

$$\frac{\partial \ln L(\theta_1, \dots, \theta_n)}{\partial \theta_i} = 0, \quad i = 1, 2, \dots, k, \text{ solve } k \text{ eq. for } k \text{ unknown.}$$

اذا كان التوزيع يحوي معلمتان فيتم الاشتقاق مرتين مرة للمعلمة (1) ومرة للمعلمة (2)

Ex1 / let $X_i, i = 1, 2, \dots, n$ be r.v.s from distribution having pdf given as. $f(x) = (1-\theta)^{x-1}\theta, \; x = 1, 2, \dots, n$

Estimate θ by MLE method.

Sol/

نقوم بإيجاد دالة الإمكان من خلال مايلي

$$L(\theta) = f(x_1) \cdot f(x_2) \cdots f(x_n)$$

$$= (1-\theta)^{x_1-1}\theta \cdot (1-\theta)^{x_2-1}\theta \cdots (1-\theta)^{x_n-1}\theta = \pi(1-\theta)^{x_i}$$

$$= \theta^n (1-\theta)^{\sum_{i=1}^{n}(x_i-1)}$$

مشتقة داخل القوس

$$\ln(1-\theta) = \frac{1}{1-\theta} \cdot (-1)$$

توضيح.

$$\ln L(\theta) = n \ln \theta + (\sum (x_i - 1)) \ln (1-\theta)$$

$$= \frac{n}{\hat{\theta}} + \frac{\sum_{i=1}^{n}(x_i-1)}{1-\hat{\theta}}(-1) = 0$$

$$\frac{n}{\hat{\theta}} + \frac{\sum(x_i-1)}{1-\hat{\theta}}$$

$$\sum(x_i - 1) = S$$

$$\frac{n}{\hat{\theta}} = \frac{S}{1-\hat{\theta}}$$

$$n - n\hat{\theta} = \hat{\theta} S$$

$$\frac{n}{\hat{\theta}} - n = S$$

$$\text{So then } \hat{\theta} = \frac{n}{n + \sum_{i=1}^{n}(x_i - 1)} = \frac{n}{\sum_{i=1}^{n} x_i}$$

$$\frac{n}{\hat{\theta}} \Rightarrow n = \hat{\theta}(S+n) \div S+n \Rightarrow \hat{\theta} = \frac{n}{n+S} \Rightarrow = \frac{n}{n + \sum(x_i - 1)}$$

Ex2/ let X be normally dist. with mean μ and variance σ^2 where both μ and σ^2 are unknown The likelihood function for a random sample of size n is.

Sol/

$$L(\mu,\sigma^2)=\prod_{i=1}^{n}\left[\frac{1}{\sigma\sqrt{2\pi}}e^{-\frac{1}{2}\frac{(x_i-\mu)^2}{\sigma^2}}\right]=\frac{1}{(2\pi\sigma^2)^{\frac{n}{2}}}e^{-\frac{1}{2\sigma^2}\Sigma(x_i-\mu)^2}$$

$$\ln L(\mu,\sigma^2)=\ln\left[(2\pi\sigma^2)^{-\frac{n}{2}}e^{-\frac{1}{2\sigma^2}\Sigma(x_i-\mu)^2}\right]$$

$$=-\frac{n}{2}\ln(2\pi\sigma^2)-\frac{1}{2\sigma^2}\sum_{i=1}^{n}(x_i-\mu)^2$$

$$\frac{\partial\ln(L)}{\partial\mu}=-\frac{1}{\cancel{2}\sigma^2}\Sigma\cancel{2}(x_i-\mu)\cdot(-1)=\frac{1}{\sigma^2}\Sigma(x_i-\mu)$$

$$=\frac{1}{\sigma^2}\Sigma(x_i-\hat{\mu})=0\ ]\cdot\sigma^2\Rightarrow\Sigma(x_i-\hat{\mu})=0$$

$$=\Sigma x_i-\Sigma\hat{\mu}=0\Rightarrow\Sigma x_i=n\hat{\mu}\Rightarrow\hat{\mu}=\frac{\Sigma x_i}{n}=\bar{x}$$

$$\frac{\partial\ln(L)}{\partial\sigma^2}=-\frac{n}{2}(\ln(2\pi)+\ln\sigma^2)+\frac{S}{2}(\sigma^2)^{-1}\qquad\text{where } S=\Sigma(x_i-\mu)^2$$

$$=\frac{-n}{2}\cdot\frac{1}{\sigma^2}+\frac{S}{2\sigma^4}\Rightarrow\frac{-n}{2\sigma^2}+\frac{S}{2\sigma^4}\ ]\cdot\sigma^4$$

$$\frac{-n\hat{\sigma}^2}{2}+\frac{S}{2}=0\Rightarrow-\frac{n\sigma^2}{2}=-\frac{S}{2}\ ]\cdot\frac{-2}{n}$$

$$\hat{\sigma}^2=\frac{S}{n}\Rightarrow\frac{1}{n}\Sigma(x_i-\hat{\mu})^2$$

$$=\frac{1}{n}\Sigma(x_i-\bar{x})^2$$

EX3/ let $f(x,\theta) = e^{-(x-\theta)}$ $\quad \theta \le x < \infty$ Estimate θ by MLE method.

Sol/

$$L(\theta) = f(x_1) \cdot f(x_2) \cdots f(x_n)$$

$$= e^{-(x_1-\theta)} \cdot e^{-(x_2-\theta)} \cdots e^{-(x_n-\theta)} = \exp\left(n\theta - \sum_{i=1}^{n} x_i\right)$$

?

$$\ln L(\theta) = n\theta - \sum_{i=1}^{n} x_i$$ ← عند الاشتقاق الناتج المتبقي هو n فقط

يلاحظ هنا عدم امكانية ايجاد مقدر الامكان الاعظم للمعلمة ويلاحظ
ايضاً ان مجال المتغير يعتمد على المعلمة ومنه فان مقدر المعلمة سيكون
من خلال الاحصاءات المرتبة وتكون $\hat{\theta} = Y_1 = \min(X_i)$

EX4/ let $X_i \sim P(\theta)$ $i = 1, 2, \ldots, n$ Estimate θ by MLE Method

Sol/

$$L(\theta) = f(x_1) \cdot f(x_2) \cdots f(x_n)$$

Since $X \sim P(\theta) \Rightarrow f(x) = \dfrac{e^{-\theta} \cdot \theta^{x}}{x!}$

$$\therefore L(\theta) = \frac{e^{-n\theta} \cdot \theta^{\sum x_i}}{\left(\prod_{i=1}^{n} x_i\right)!}$$

$$\ln L(\theta) = -n\theta + \left(\sum_{i=1}^{n} x_i\right) \ln(\theta) - \ln\left(\prod_{i=1}^{n} x_i!\right)$$

$$\frac{\partial \ln L(\theta)}{\partial \theta} = -n + \frac{\sum x_i}{\hat{\theta}} \Rightarrow 0$$

$$-n + \frac{\sum x_i}{\hat{\theta}} = 0 \Rightarrow n = \frac{\sum x_i}{\hat{\theta}} \Rightarrow \hat{\theta} = \frac{\sum x_i}{n} = \bar{x}$$

is The MLE estimator for θ.