Non-homogeneous linear differential equation with constant coefficient

Short method for finding particular integral (undetermined coefficient)

let $L(D) = f(x)$ then

$f(x)$	y_p
1- ae^{bx}	Ae^{bx}
2- $\mathrm{a}\cos(bx)$ or $\mathrm{a}\sin(bx)$	$\mathrm{A}\cos(bx) + \mathrm{B}\sin(bx)$
3- $a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0$	$Ax^n + Bx^{n-1} + \cdots + C$
4- $e^{ax}\cos bx$ or $e^{ax}\sin bx$	$e^{ax}(\mathrm{A}\cos bx + B\sin bx)$
5- $e^{ax}(a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0)$	$e^{ax}(Ax^n + Bx^{n-1} + \cdots + C)$

Note: - for repeat term (root) multiply by x .

Examples: find the general solution of the ODE

1- $(D^3 - 3D^2 + 3D - 1)y = 4x^3 - 2x - 4$

$(D^3 - 3D^2 + 3D - 1)y = 0$

The characteristic equation is

$m^3 - 3m^2 + 3m - 1 = 0$

$m_1 = 1 \quad \rightarrow \quad m - 1 = 0$

$(m^2 - 2m + 1) = 0$

$(m - 1)(m - 1) = 0$

$m_2 = 1 \ , \ m_3 = 1$

$y_c = c_1 e^{m_1 x} + c_2 x e^{m_2 x} + c_3 x^2 e^{m_3 x}$

$y_c = c_1 e^x + c_2 x e^x + c_3 x^2 e^x$

$y_1 = 1, \quad y_2 = xe^x, \quad y_3 = x^2 e^x$

To find $y_p \rightarrow f(x) = 4x^3 - 2x - 4$

$y_p = Ax^3 + Bx^2 + Cx + D$

$y_p' = 3Ax^2 + 2Bx + C$

$y_p'' = 6Ax + 2B$

$y_p''' = 6A$

$6A - 3(6Ax + 2B) + 3(3Ax^2 + 2Bx + C) - (Ax^3 + Bx^2 + Cx + D) = 4x^3 - 2x - 4$

$6A - 18Ax - 6B + 9Ax^2 + 6Bx + 3C - Ax^3 - Bx^2 - Cx - D = 4x^3 - 2x - 4$

$-A = 4 \quad \rightarrow \quad A = -4$

$9A - B = 0 \quad \rightarrow \quad B = -36$

$-18A + 6B - C = -2 \quad \rightarrow \quad C = -142$

$6A - 6B + 3C + D = -4 \quad \rightarrow \quad D = -662$

$y_p = -4x^3 - 36x^2 - 142x - 662$

the general solution $y = y_c + y_p$

$y = c_1e^x + c_2xe^x + c_3x^2e^x - 4x^3 - 36x^2 - 142x - 662$

2- $(D^4 - 9)y = 3e^x$

$(D^4 - 9)y = 0$

The characteristic equation is

$m^4 - 9 = 0$

$(m^2 - 3)(m^2 + 3) = 0$

$m_{1,2} = \mp\sqrt{3},\ m_{3,4} = \mp\sqrt{3}i$

$y_c = c_1e^{m_1x} + c_2e^{m_2x} + e^{ax}(c_3 \cos bx + c_4 \sin bx)$

$y_c = c_1e^{\sqrt{3}x} + c_2e^{-\sqrt{3}x} + c_3 \cos\sqrt{3}x + c_4 \sin\sqrt{3}x$

$y_1 = e^{\sqrt{3}x}, \quad y_2 = xe^{-\sqrt{3}x}, \ y_3 = \cos\sqrt{3}x, \quad y_4 = \sin\sqrt{3}x$

To find $y_p \rightarrow f(x) = 3e^x$

$y_p = Ae^x$, $y_p' = Ae^x$, $y_p'' = Ae^x$, $y_p''' = Ae^x$, $y_p'''' = Ae^x$

$Ae^x - 9Ae^x = 3e^x \ \rightarrow \ -8Ae^x = 3e^x \ \rightarrow A = \frac{-3}{8}$

$y_p = \frac{-3}{8} e^x$

the general solution $\ y = y_c + y_p$

$y = c_1 e^{\sqrt{3}x} + c_2 e^{-\sqrt{3}x} + c_3 \cos\sqrt{3}x + c_4 \sin\sqrt{3}x - \frac{3}{8} e^x$

3- $(D^2 + 2D + 2)y = \sin x$

$(D^2 + 2D + 2)y = 0$

The characteristic equation is

$m^2 + 2m + 2 = 0$

$m_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \quad \rightarrow \quad m_{1,2} = \frac{-2 \pm \sqrt{4-8}}{2}$

$m_{1,2} = \frac{-2 \pm \sqrt{-4}}{2} \quad \rightarrow \quad m_{1,2} = \frac{-2 \pm 2i}{2}$

$m_{1,2} = -1 \pm i \quad m_1 = -1 + i \ , \ m_2 = -1 - i$

$y_c = e^{ax}(c_1 \cos bx + c_2 \sin bx)$

$y_c = e^{-x}(c_1 \cos x + c_2 \sin x)$

$y_1 = e^{-x} c_1 \cos x \quad , \quad y_2 = e^{-x} c_2 \sin x$

To find $y_p \ \rightarrow \ f(x) = \sin x$

$y_p = \text{A} \cos x + B \sin x$

$y_p' = -\text{A} \sin x + B \cos x$

$y_p'' = -\text{A} \cos x - B \sin x$

$-\text{A} \cos x - B \sin x + 2(-\text{A} \sin x + B \cos x) + 2(\text{A} \cos x + B \sin x) = \sin x$

$-\text{A} \cos x - B \sin x - 2\text{A} \sin x + 2B \cos x + 2\text{A} \cos x + 2B \sin x = \sin x$

$-A + 2B + 2A = 0 \ \rightarrow \ A + 2B = 0 \ldots\ldots\ldots (1)$

$-B - 2A + 2B = 1 \ \rightarrow \ -2A + B = 1 \ldots\ldots (2)$

$A = \frac{-2}{5}$, $B = \frac{1}{5}$

$y_p = \frac{-2}{5}\cos x + \frac{1}{5}\sin x$

the general solution $y = y_c + y_p$

$y = e^{-x}(c_1 \cos x + c_2 \sin x) - \frac{2}{5}\cos x + \frac{1}{5}\sin x$

4- $(D^3 - D^2 - D + 1)y = x^2 + x + 3e^{-x} - 2\cos 2x$

$(D^3 - D^2 - D + 1)y = 0$

The characteristic equation is

$m^3 - m^2 - m + 1 = 0$

$m_1 = 1 \rightarrow m - 1 = 0$

$(m^2 - 1) = 0$

$m_2 = 1$, $m_3 = -1$

$y_c = c_1 e^{m_1 x} + c_2 x e^{m_2 x} + c_3 e^{m_3 x}$

$y_c = c_1 e^{x} + c_2 x e^{x} + c_3 e^{-x}$

$y_1 = e^x, \quad y_2 = xe^x, \quad y_3 = e^{-x}$

To find $y_p \rightarrow f(x) = x^2 + x + 3e^{-x} - 2\cos 2x$

$f_1(x) = x^2 + x \rightarrow y_{p1} = Ax^2 + Bx + C$

$y'_{p1} = 2Ax + B \rightarrow y''_{p1} = 2A \rightarrow y'''_{p1} = 0$

$0 - 2A - 2Ax - B + Ax^2 + Bx + C = x^2 + x$

$A = 1 \rightarrow -2A + B = 1 \rightarrow B = 3$

$-2A - B + C = 0 \rightarrow C = 5$

$y_{p1} = x^2 + 3x + 5$

$f_2(x) = 3e^{-x} \rightarrow y_{p2} = Axe^{-x}$

$y'_{p2} = -Axe^{-x} + Ae^{-x}$

$y''_{p2} = Axe^{-x} - Ae^{-x} - Ae^{-x} \rightarrow y''_{p2} = Axe^{-x} - 2Ae^{-x}$

$y_{p1}''' = -Axe^{-x} + Ae^{-x} + 2Ae^{-x} \rightarrow y_{p1}''' = -Axe^{-x} + 3Ae^{-x}$

$-Axe^{-x} + 3Ae^{-x} - Axe^{-x} + 2Ae^{-x} + Axe^{-x} - Ae^{-x} +$

$Axe^{-x} = 3e^{-x}$

$4A = 3 \rightarrow A = \frac{3}{4} \rightarrow y_{p2} = \frac{3}{4}xe^{-x}$

$f_3(x) = -2\cos 2x \quad \rightarrow y_{p3} = \mathrm{A}\cos 2x + B\sin 2x$

$y_{p3}' = -2\mathrm{A}\sin x + 2B\cos x$

$y_{p3}'' = -4\mathrm{A}\cos 2x - 4B\sin 2x$

$y_{p3}''' = 8\mathrm{A}\sin x - 8B\cos x$

$8\mathrm{A}\sin x - 8B\cos x + 4\mathrm{A}\cos 2x + 4B\sin 2x +$

$2\mathrm{A}\sin x - 2B\cos x + \mathrm{A}\cos 2x + B\sin 2x$

$-8B + 4A - 2B + A = -2 \quad \rightarrow \quad 5A - 10B = -2 \ldots\ldots (1)$

$8A + 4B + 2A + B \qquad\qquad \rightarrow \quad 10A + 5B = 0 \ldots\ldots\ldots (2)$

$A = \frac{-2}{15} \quad , \quad B = \frac{4}{15} \quad \rightarrow y_{p3} = \frac{-2}{15}\cos 2x + \frac{4}{15}\sin 2x$

the general solution $\quad y = y_c + y_{p1} + y_{p2} + y_{p3}$

$y = c_1e^x + c_2xe^x + c_3e^{-x} + x^2 + 3x + 5 + \frac{3}{4}xe^{-x} -$

$\frac{2}{15}\cos 2x + \frac{4}{15}\sin 2x$

5- $(D^2 + 5D)y = (x-1)e^x$

$(D^2 + 5D)y = 0$

The characteristic equation is

$m^2 + 5m = 0$

$m(m+5) = 0$

$m_1 = 0 \;,\; m_2 = -5 \;,$

$y_c = c_1e^{m_1x} + c_2e^{m_2x}$

$y_c = c_1e^{0x} + c_2e^{-5x}$

$y_1 = 1 \;, \quad y_2 = e^{-5x}$

To find $y_p \;\to\; f(x) = (x-1)e^x = xe^x - e^x$

$y_p = Axe^x + Be^x \;\to\; y_p' = Axe^x + Ae^x + Be^x$

$y_p'' = Axe^x + Ae^x + Ae^x + Be^x \to y_p'' = Axe^x + 2Ae^x + Be^x$

$Axe^x + 2Ae^x + Be^x + 5Axe^x + 5Ae^x + 5Be^x = xe^x - e^x$

$6A = 1 \;\to\; A = \frac{1}{6}$

$7A + 6B = -1 \;\to\; B = \frac{-13}{36}$

$y_p = \frac{1}{6}xe^x - \frac{13}{36}e^x$

the general solution $y = y_c + y_p$

$y = c_1 + c_2e^{-5x} + \frac{1}{6}xe^x - \frac{13}{36}e^x$

Method of variation of parameterizes

Examples: find the general solution of the ODE

1- $(D^2+1)y = \sec x$

$(D^2+1)y = 0$

The characteristic equation is

$m^2 + 1 = 0$

$m_{1,2} = \mp i$

$y_c = e^{ax}(c_1\cos bx + c_2\sin bx)$

$y_c = c_1\cos x + c_2\sin x$

$y_1 = \cos x, \quad y_2 = \sin x$

To find $y_p \;\to\; f(x) = \sec x$

$y_p = N_1y_1 + N_2y_2$

$w(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix} = \cos^2 x + \sin^2 x = 1$

$N_1 = -\int \frac{y_2.f(x)}{w} = -\int \frac{\sin x.\sec x}{1} = -\int \frac{\sin x}{\cos x} = ln|\cos x|$

$N_2 = \int \frac{y_1.f(x)}{w} = \int \frac{\cos x.\sec x}{1} = -\int \frac{\cos x}{\cos x} = x$

$y_p = \cos x \, ln|\cos x| \; + x \sin x$

the general solution $\quad y = y_c + y_p$

$y = c_1 \cos x + c_2 \sin x + \cos x \, ln|\cos x| \; + x \sin x$

2- $(D^2 - 6D + 9)y = \frac{e^{3x}}{x^2}$

$(D^2 - 6D + 9)y = 0$

The characteristic equation is

$m^2 - 6m + 9 = 0$

$(m-3)(m-3) = 0$

$m_1 = 3 \;\;, \; m_2 = 3$

$y_c = c_1 e^{m_1 x} + c_2 x e^{m_2 x}$

$y_c = c_1 e^{3x} + c_2 x e^{3x}$

$y_1 = e^{3x}, \quad y_2 = x e^{3x}$

To find $y_p \;\; \rightarrow \; f(x) = \frac{e^{3x}}{x^2}$

$y_p = N_1 y_1 + N_2 y_2$

$w(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} e^{3x} & x e^{3x} \\ 3e^{3x} & 3x e^{3x} + e^{3x} \end{vmatrix}$

$= 3x e^{6x} + e^{6x} - 3x e^{6x} = e^{6x}$

$N_1 = -\int \frac{y_2.f(x)}{w} = -\int \frac{x e^{3x}.\frac{e^{3x}}{x^2}}{e^{6x}} = \int \frac{1}{x} = -ln|x|$

$N_2 = \int \frac{y_1.f(x)}{w} = \int \frac{e^{3x}.\frac{e^{3x}}{x^2}}{e^{6x}} = \int \frac{1}{x^2} = -\frac{1}{x}$

$y_p = -e^{3x} ln|x| - e^{3x}$

the general solution $\quad y = y_c + y_p$

$y = c_1 e^{3x} + c_2 x e^{3x} - e^{3x} ln|x| - e^{3x}$

H.W. Find the general solution of the differential equation

1-- $(D^2+2D-3)y=(x+1)^2e^{x^2}$

2-- $(D^2+D-2)y=3e^{-x}+1$

3-- $(D^2+4D+4)y=x^2-e^{-2x}$

4-- $(D^2-2D+1)y=\frac{e^x}{(1-x)^2}$

5-- $(D^2-2D+2)y=\sin x+3\cos x$

6-- $(D^2-2D+2)y=x(\sin x+1)$

7-- $(D^2-4D+4)y=x^2\ln(x)$

8-- $(D^3-5D^2-7D-1)y=e^{-3x}\cosh x$

9-- $(D^3-3D^2+4)y=2-3x+x^2$

10-- $(D^3+2D^2+D)y=3e^{-x}-2x^2-x+4\sin 3x$

11-- $(D^3-3D-2)y=e^x\cos x$

12-- $(2D+1)^2y=4e^{\frac{-x}{2}}$

13-- $(D^4+3D^2-4)y=(e^{2x}+3)^2$

14-- $(D^3-4D^2+2D-8)y=e^x(\cos 2x+x^2)-6$

15-- $(4D^3+16D^2+21D+9)y=\cosh 2x$

16-- $(D^2-13D-12)y=\tan 5x$

17-- $(9D^2-12D+4)y=e^{\frac{-x}{4}}+e^{-x}+\sqrt{2x+1}$

18--$(D^3+6D^2+3D-10)y=(4-x^2)^2$