linear differential equation with variable coefficient(Euler – Cauchy equation)

it has form: -

$$a_n x^n \frac{d^n y}{dx^n} + a_{n-1} x^{n-1} \frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_0 y = f(x) \ldots\ldots\ldots\ldots\ldots\ldots. (1)$$

or $(a_n x^n D^n + a_{n-1} x^{n-1} D^{n-1} + \cdots + a_0) y = 0 \ldots\ldots\ldots\ldots\ldots\ldots. (2)$

we can write as the form $\left(\sum_{r=0}^{n} a_r x^{n-r} D^{n-1}\right) y = f(x) \ldots\ldots\ldots (3)$

where a_r are constant, $f(x)$ is function of x is called an Euler – Cauchy equation. to solve this equation, we put

let $x = e^z \;\; \rightarrow \;\; z = \ln(x) \;\; then \; \frac{dz}{dx} = \frac{1}{x}$

$$\frac{dy}{dx} = \frac{dy}{dz} \cdot \frac{dz}{dx} \;\; \rightarrow \;\; \frac{dy}{dx} = \frac{dy}{dz} \cdot \frac{1}{x} \;\; \rightarrow x \frac{dy}{dx} = \frac{dy}{dz} = dy \;\; \ldots\ldots\ldots\ldots\ldots\ldots (4)$$

$$\frac{d^2 y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left(\frac{dy}{dz} \cdot \frac{1}{x}\right) = \frac{1}{x^2}\left(\frac{d^2 y}{dz^2} - \frac{dy}{dz}\right)$$

$$x^2 \frac{d^2 y}{dx^2} = \frac{d^2 y}{dz^2} - \frac{dy}{dz} \;\; or \;\; x^2 \frac{d^2 y}{dx^2} = d^2 y - dy = d(d-1)y \ldots\ldots. (5)$$

By induction $x^n D^n y = d(d-1)(d-2) \ldots\ldots (d - r + 1) \ldots\ldots (6)$

Where $dy = \frac{dy}{dz}$

Examples: find the general solution of the ODE

1- $(4x^2 D^2 - 3xD + 3)y = 0$

   let $x = e^z \;\; \rightarrow \;\; z = \ln(x) \;\; then \; \frac{dz}{dx} = \frac{1}{x}$

   $xD = d \;\; and \;\; x^2 D^2 = d(d-1) = d^2 - d$

   $[4(d^2 - d) - 3d + 3]y = 0$

   $(4d^2 - 7d + 3)y = 0$

   $4m^2 - 7m + 3 = 0$

   $(4m - 3)(m - 1) = 0$

$4m - 3 = 0 \quad \rightarrow \quad m_1 = \frac{3}{4}$

$m - 1 = 0 \quad \rightarrow \quad m_2 = \frac{1}{2}$

the general solution $y = c_1 e^{m_1 z} + c_2 e^{m_2 z}$

$y = c_1 e^{\frac{3}{4}z} + c_2 e^{z} \quad \rightarrow \quad y = c_1 e^{\frac{3}{4}\ln(x)} + c_2 e^{\ln(x)}$

$y = c_1 x^{\frac{3}{4}} + c_2 x \qquad y_1 = x^{\frac{3}{4}} \;, \quad y_2 = x$

2- $(x^2 D^2 + 5xD + 4)y = \ln x^2$

let $x = e^z \quad \rightarrow \quad z = \ln(x) \;\; then \; \frac{dz}{dx} = \frac{1}{x}$

$xD = d \;\; and \;\; x^2 D^2 = d(d-1) = d^2 - d$

$(d^2 - d + 5d + 4)y = 0$

$(d^2 + 4d + 4)y = 0$

$m^2 + 4m + 4 = 0$

$(m+2)(m+2) = 0$

$m_1 = m_2 = -2$

$y_c = c_1 e^{m_1 z} + c_2 z e^{m_2 z} \qquad y_c = c_1 e^{-2z} + c_2 z e^{-2z}$

$y_c = c_1 e^{-2\ln(x)} + c_2 \ln(x)\, e^{-2\ln(x)}$

$y_c = c_1 \frac{1}{x^2} + c_2 \frac{\ln(x)}{x^2} \qquad y_1 = \frac{1}{x^2} \;, \quad y_2 = \frac{\ln(x)}{x^2}$

To find $y_p \;\; \rightarrow \; f(z) = 2z$

$y_p = Az + B \quad y_p' = A \quad y_p'' = 0$

$0 + 4A + 4Az + B = 2z$

z: $4A = 2 \quad \rightarrow A = \frac{1}{2}$

C: $4A + B = 0 \quad \rightarrow \quad B = \frac{-1}{2}$

$y_p = \frac{1}{2} z - \frac{1}{2} \qquad y_p = \frac{1}{2}\ln(x) - \frac{1}{2}$

the general solution $\;\; y = y_c + y_p$

$y = c_1 \frac{1}{x^2} + c_2 \frac{\ln(x)}{x^2} + \frac{1}{2}\ln(x) - \frac{1}{2}$

3- $(x^3D^3 + 2x^2D^2 + 2)y = 10(x + \frac{1}{x})$

let $x = e^z \;\; \rightarrow \;\; z = \ln(x) \;\; then \; \frac{dz}{dx} = \frac{1}{x}$

$xD = d \;,\; x^2D^2 = d(d-1) = d^2 - d$

$x^3D^3 = d(d-1)(d-2) = d^3 - 3d^2 + 2d$

$(d^3 - 3d^2 + 2d + 2d^2 - 2d + 2)y = 10e^z + 10e^{-z}$

$(d^3 - d^2 + 2)y = 10e^z + 10e^{-z}$

$m^3 - m^2 + 2 = 0$

$m_1 = -1 \;\; \rightarrow \;\; m + 1 = 0$

$m^2 - 2m + 2 = 0$

$m_{2,3} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \;\; \rightarrow \qquad m_{2,3} = \frac{2 \pm \sqrt{4-8}}{2}$

$m_{2,3} = \frac{2 \pm \sqrt{-4}}{2} \qquad \rightarrow \qquad m_{2,3} = \frac{2 \pm 2i}{2}$

$m_{2,3} = 1 \pm i \qquad m_2 = 1 + i \;\;,\;\; m_3 = 1 - i$

$y_c = c_1 e^{m_1 z} + e^{az}(c_2 \cos bz + c_2 \sin bz)$

$y_c = c_1 e^{-z} + e^z(c_2 \cos z + c_3 \sin z)$

$y_c = c_1 e^{-\ln(x)} + e^{\ln(x)}(c_2 \cos \ln(x) + c_3 \sin \ln(x))$

$y_c = \frac{c_1}{x} + x(c_2 \cos \ln(x) + c_3 \sin \ln(x))$

$y_2 = \frac{1}{x} \;\;,\;\; y_2 = x\cos[\ln(x)] \quad , \quad y_3 = x \sin[\ln(x)]$

To find $y_p \;\; \rightarrow \;\; f(z) = 10e^z + 10e^{-z}$

$y_p = Ae^z + Bze^{-z}$

$y_p' = Ae^z - Bze^{-z} + Be^{-z}$

$y_p'' = Ae^z + Bze^{-z} - Be^{-z} - Be^{-z}$

$y_p'' = Ae^z + Bze^{-z} - 2Be^{-z}$

$y_p''' = Ae^z - Bze^{-z} + Be^{-z} + 2Be^{-z}$

$y_p''' = Ae^z - Bze^{-z} + 3Be^{-z}$

$Ae^z - Bze^{-z} + 3Be^{-z} - Ae^z - Bze^{-z} + 2Be^{-z} + 2Ae^z +$
$2Bze^{-z} = 10e^z + 10e^{-z}$

$2Ae^{z} + 5Be^{-z} = 10e^{z} + 10e^{-z}$

$e^{z}: 2A = 10 \quad \rightarrow A = 5$

$e^{z}: 5B = 10 \quad \rightarrow B = 2$

$y_p = 5e^{z} + 2ze^{-z} \quad \rightarrow \quad y_p = 5e^{\ln(x)} + 2\ln(x)\, e^{-\ln(x)}$

$y_p = 5x + \frac{2\ln(x)}{x}$

the general solution $\quad y = y_c + y_p$

$y = \frac{c_1}{x} + x(c_2 \cos\ln(x) + c_3 \sin\ln(x)) + 5x + \frac{2\ln(x)}{x}$

4- $(x^3D^3 + 3x^2D^2 - 2xD + 2)y = \cos(lnx^2)$

let $x = e^{z} \quad \rightarrow \quad z = \ln(x) \quad then\ \frac{dz}{dx} = \frac{1}{x}$

$xD = d \ ,\ x^2D^2 = d(d-1) = d^2 - d$

$x^3D^3 = d(d-1)(d-2) = d^3 - 3d^2 + 2d$

$(d^3 - 3d^2 + 2d + 3d^2 - 3d - 2d + 2)y = \cos 2z$

$(d^3 - d + 2)y = \cos 2z$

$m^3 - 3m + 2 = 0$

$m_1 = 1 \quad \rightarrow \quad m - 1 = 0$

$m^2 + m - 2 = 0$

$(m+2)(m-1) = 0$

$m_2 = 1 \qquad m_3 = -2$

$y_c = c_1e^{m_1z} + c_2ze^{m_2z} + c_3e^{m_3z}$

$y_c = c_1e^{z} + c_2ze^{z} + c_3e^{-2z}$

$y_c = c_1x + c_2xlnx + \frac{c_3}{x^2}$

To find $y_p \quad \rightarrow \quad f(z) = \cos 2z$

$y_p = \mathrm{A}\cos 2z + B\sin 2z$

$y_p' = -2\mathrm{A}\sin 2z + 2B\cos 2z$

$y_p'' = -4\mathrm{A}\cos 2z - 4B\sin 2z$

$y_p''' = 8\mathrm{A}\sin 2z - 8B\cos 2z$

$8A\sin 2z - 8B\cos 2z + 6A\sin 2z - 6B\cos 2z + 2A\cos 2z +$
$2B\sin 2z = \cos 2z$

$\cos 2z: -8B - 6B + 2A = 1 \;\; \rightarrow \;\; -14B + 2A = 1 \; \ldots\ldots (1)$

$\sin 2z: 8A + 6A + 2B = 0 \;\; \rightarrow \;\; 2B + 14A = 0 \ldots\ldots\ldots (2)$

$A = \frac{1}{100} \;\; , B = \frac{-7}{100}$

$y_p = \frac{1}{100}\cos 2z - \frac{7}{100}\sin 2z$

$y_p = \frac{1}{100}\cos 2\ln(x) - \frac{7}{100}\sin 2\ln(x)$

the general solution $\;\; y = y_c + y_p$

$y = c_1 x + c_2 x lnx + \frac{c_3}{x^2} + \frac{1}{100}\cos 2\ln(x) - \frac{7}{100}\sin 2\ln(x)$

H.W. Find the general solution of the differential equation

1-- $x^2y'' - 7xy' + 16y = x^4 lnx$
2-- $x^3y''' + 2xy' - 2y = \sin lnx$
3-- $(x^2D^2 - 3xD + 4)y = 3x^2 lnx + 2$
4-- $x^2y'' + xy' - 9y = x^3 - 1$
5-- $(x^3D^3 - 3x^2D^2 + 6xD - 6)y = 0$
6-- $(2x^2D^2 + 5xD + 1)y = \frac{6}{x^3}$