Laplace transformation

Give a well define function $f(x)$ the laplace transform for function $f(x)$ written $\mathcal{L}\{f(x)\}$ or $F(s)$ is define by

$\mathcal{L}\{f(x)\} = F(s) = \int_0^\infty f(x).e^{-sx}\,dx \qquad x > 0 \quad \ldots\ldots\ldots(1)$

Where (s) is real or complex number

Note:- condition for Laplace transform of the $f(x)$

1- The integral must be convergent

2- $f(x)$ is piece wise continuous and exponential of x

Laplace transformation for some elementry function

1- $f(x) = 1$

$\mathcal{L}\{1\} = \int_0^\infty 1.e^{-sx}\,dx = \frac{-1}{s}e^{-sx}\Big]_0^\infty = 0 + \frac{1}{s} = \frac{1}{s}$

2- $f(x) = x$

$\mathcal{L}\{x\} = \int_0^\infty x.e^{-sx}\,dx = \frac{-x}{s}e^{-sx} - \frac{1}{s^2}e^{-sx}\Big]_0^\infty = \frac{1}{s^2}$

3- $f(x) = e^{ax}$

$\mathcal{L}\{e^{ax}\} = \int_0^\infty e^{ax}.e^{-sx}\,dx = \int_0^\infty e^{ax-sx}\,dx = \int_0^\infty e^{-(s-a)x}\,dx$

$= \frac{-1}{s-a}e^{-(s-a)x}\Big]_0^\infty = \frac{1}{s-a}$

4- $f(x) = \cos ax$

$\mathcal{L}\{\cos ax\} = \int_0^\infty \cos ax\,.e^{-sx}\,dx = \frac{s}{s^2+a^2}$

5- $f(x) = \sin ax$

$\mathcal{L}\{\sin ax\} = \int_0^\infty \sin ax\,.e^{-sx}\,dx = \frac{a}{s^2+a^2}$

Examples: find Laplace transform

a) $\mathcal{L}\{\cos 4x\} = \frac{s}{s^2+16}$

b) $\mathcal{L}\{\sin 3x\} = \frac{3}{s^2+9}$

Some property of Laplace transformation

1- **Linearity**:- let c_1, c_2 constant then
$\mathcal{L}\{c_1 f(x)_1 + c_2 f(x)_2\} = c_1\mathcal{L}\{f(x)_1\} + c_2\mathcal{L}\{f(x)_2\}$
Proof:-
$\mathcal{L}\{c_1 f(x)_1 + c_2 f(x)_2\} = \int_0^\infty [c_1 f(x)_1 + c_2 f(x)_2].e^{-sx}\, dx$
$= \int_0^\infty c_1 f(x)_1.e^{-sx}\, dx + \int_0^\infty c_2 f(x)_2].e^{-sx}\, dx$
$= c_1\int_0^\infty f(x)_1.e^{-sx}\, dx + c_2\int_0^\infty f(x)_2].e^{-sx}\, dx$
$= c_1\mathcal{L}\{f(x)_1\} + c_2\mathcal{L}\{f(x)_2\}$

a) $\mathcal{L}\{\sin x + 5e^{-3x} + 2\} = \mathcal{L}\{\sin x\} + \mathcal{L}\{5e^{-3x}\} + \mathcal{L}\{2\}$
$= \frac{1}{s^2+1} + \frac{5}{s+3} + \frac{2}{s}$

b) $\mathcal{L}\{\cosh ax\} = \mathcal{L}\{\frac{e^{ax}+e^{-ax}}{2}\} = \frac{1}{2}\mathcal{L}\{e^{ax}\} + \frac{1}{2}\mathcal{L}\{e^{-ax}\}$
$= \frac{1}{2}[\frac{1}{s-a} + \frac{1}{s+a}] = \frac{s}{s^2-a^2}$

c) $\mathcal{L}\{\sinh ax\} = \mathcal{L}\{\frac{e^{ax}-e^{-ax}}{2}\} = \frac{1}{2}\mathcal{L}\{e^{ax}\} - \frac{1}{2}\mathcal{L}\{e^{-ax}\}$
$= \frac{1}{2}[\frac{1}{s-a} - \frac{1}{s+a}] = \frac{a}{s^2-a^2}$

2- **Shifting**:- $\mathcal{L}\{e^{ax}.f(x)\} = F(s-a)$

a) $\mathcal{L}\{e^{2x}.x^3\} \rightarrow \mathcal{L}\{x^3\} = \frac{6}{s^2} \rightarrow \mathcal{L}\{e^{2x}.x^3\} = \frac{6}{(s-2)^2}$
b) $\mathcal{L}\{e^{-2x}.\sin 4x\} \rightarrow \mathcal{L}\{\sin 4x\} = \frac{4}{s^2+16}$
$\rightarrow \mathcal{L}\{e^{-2x}.\sin 4x\} = \frac{4}{(s+2)^2+16}$

3- $\boldsymbol{\mathcal{L}\{x^n.f(x)\} = (-1)^n \frac{d^n}{ds^n}\mathcal{L}\{f(x)\}}$

a) $\mathcal{L}\{x.\cosh 3x\} = (-1)^1 \frac{d}{ds}\mathcal{L}\{\cosh 3x\}$
$= (-1)^1 \frac{d}{ds}(\frac{s}{s^2-9}) = \frac{s^2+9}{(s^2-9)^2}$

b) $\mathcal{L}\{x^2.e^{-2x}\} = (-1)^2\frac{d^2}{ds^2}\mathcal{L}\{e^{-2x}\}$

$= (-1)^2\frac{d^2}{ds^2}\mathcal{L}\{\frac{1}{s+2}\} = \frac{d}{ds}[\frac{-1}{(s+2)^2}] = \frac{2}{(s+2)^3}$

4- **Differential** :-

a) $\mathcal{L}\{y'\} = s\mathcal{L}\{y\} - y_{(0)}$

b) $\mathcal{L}\{y''\} = s^2\mathcal{L}\{y\} - sy_{(0)} - y'_{(0)}$

c) $\mathcal{L}\{y'''\} = s^3\mathcal{L}\{y\} - s^2y_{(0)} - sy_{(0)} - y''_{(0)}$

5- **Integrating :-** $\boldsymbol{\mathcal{L}\{\int_0^x f(x)\} = \frac{1}{s}\mathcal{L}\{f(x)\}}$

a) $\mathcal{L}\{\int_0^x e^{3x}\} = \frac{1}{s}\mathcal{L}\{e^{3x}\} = \frac{1}{s}(\frac{1}{s-3}) = \frac{1}{s(s-3)}$

b) $\mathcal{L}\{\int_0^x e^x.x^3\} = \frac{1}{s}\mathcal{L}\{e^x.x^3\} = \frac{1}{s}(\frac{6}{(s-1)^4}) = \frac{6}{s(s-1)^4}$

6- **Inverse Laplace transform:-** let $\mathcal{L}\{f(x)\} = F(s)$ then $f(x) = \mathcal{L}^{-1}\{F(s)\}$ $\mathcal{L}^{-1}$is called Inverse Laplace transform

a) $\mathcal{L}^{-1}\{\frac{1}{s^2}\} = x$

b) $\mathcal{L}^{-1}\{\frac{2}{s-3}\} = 2e^{3x}$

c) $\mathcal{L}^{-1}\{\frac{4s}{s^2+16}\} = 4\mathcal{L}^{-1}\{\frac{s}{s^2+16}\} = 4\cos 4x$

d) $\mathcal{L}^{-1}\{\frac{3s-16}{s^2-64}\} = \mathcal{L}^{-1}\{\frac{3s}{s^2-64}\} - \mathcal{L}^{-1}\{\frac{16}{s^2-64}\}$

$= 3\cosh 8x - 2\sinh 8x$

e) $\mathcal{L}^{-1}\{\frac{1}{(s-5)^3}\} = \frac{1}{2}\mathcal{L}^{-1}\{\frac{2}{(s-5)^3}\} = \frac{1}{2}x^3.e^{5x}$

f) $\mathcal{L}^{-1}\{\frac{2(s+1)}{(s+1)^2+9}\} = 2\mathcal{L}^{-1}\{\frac{(s+1)}{(s+1)^2+9}\} = 2e^{-x}.\cos 3x$

g) $\mathcal{L}^{-1}\{\frac{s}{(s-3)^2}\} = \mathcal{L}^{-1}\{\frac{s-3+3}{(s-3)^2}\} = \mathcal{L}^{-1}\{\frac{1}{s-3}\} + \mathcal{L}^{-1}\{\frac{3}{(s-3)^2}\}$

$= e^{3x} + 3xe^{3x}$

h) $\mathcal{L}^{-1}\{\frac{1}{s} + \frac{2}{s(s+1)}\} = \mathcal{L}^{-1}\{\frac{1}{s}\} + \mathcal{L}^{-1}\{\frac{1}{s}.\frac{2}{s+1}\} = x + 2\int_0^x e^{-x}dx$

i) $\mathcal{L}^{-1}\{\frac{1}{s^2+4s+29}\} = \mathcal{L}^{-1}\{\frac{1}{s^2+4s+29+4-4}\} = \mathcal{L}^{-1}\{\frac{1}{s^2+4s+4+25}\}$

$\mathcal{L}^{-1}\{\frac{1}{(s+2)^2+25}\} = \frac{1}{5}\,\mathcal{L}^{-1}\{\frac{5}{(s+2)^2+25}\} = \frac{1}{5}e^{-2x}.\sin 5x$

j) $\mathcal{L}^{-1}\{\frac{2s+3}{s^2-2s+5}\} = \mathcal{L}^{-1}\{\frac{2s+3}{s^2-2s+5+1-1}\} = \mathcal{L}^{-1}\{\frac{2s+3}{(s-1)^2+4}\}$

$= \mathcal{L}^{-1}\{\frac{2s-2+5}{(s-1)^2+4}\} = \mathcal{L}^{-1}\{\frac{2(s-1)+5}{(s-1)^2+4}\}$

$= \mathcal{L}^{-1}\{\frac{2(s-1)}{(s-1)^2+4}\} + \mathcal{L}^{-1}\{\frac{5}{(s-1)^2+4}\}$

$= 2\mathcal{L}^{-1}\{\frac{(s-1)}{(s-1)^2+4}\} + \frac{5}{2}\mathcal{L}^{-1}\{\frac{2}{(s-1)^2+4}\}$

$= e^{-x}.\cos 2\mathrm{x} + \frac{2}{5}e^{-x}.\sin 2x$

H.W. Find Laplace transform for each the following equation

1-- $f(x) = 2x^2 + 3$ 2-- $f(x) = 2x^2 - 3x + 4$

3-- $f(x) = e^{-2x}.\cosh 4x$ 4-- $f(x) = x \sin x$

5-- $f(x) = 2\sin x + 3\cos 2x$ 6-- $f(x) = \int_0^x e^{-x}.x.\cos 2x \quad dx$

H.W. Evaluate the following equation

1-- $\mathcal{L}^{-1}\{\frac{1}{2s^2+1}\}$ 2-- $\mathcal{L}^{-1}\{\frac{s^3-s^2+s-1}{s^5}\}$

3-- $\mathcal{L}^{-1}\{\frac{1}{3s+4}\}$ 4-- $\mathcal{L}^{-1}\{\frac{3s+9}{s^2+2s+10}\}$

5-- $\mathcal{L}^{-1}\{\frac{2s-4}{(s-2)(s+1)(s-3)}\}$ 6-- $\mathcal{L}^{-1}\{\frac{s}{s^2-6s-7}\}$

7-- $\mathcal{L}^{-1}\{\frac{2s-6}{(2s-4)^5}\}$ 8-- $\mathcal{L}^{-1}\{\frac{1}{s^3+16}\}$