A) Using Laplace transformation to solve IVP

1- $y^{'} - 5y = 0 \qquad y_{(0)} = 2$

$\mathcal{L}\{y^{'} - 5y\} = \mathcal{L}\{0\}$

$s\mathcal{L}\{y\} - y_{(0)} - 5\mathcal{L}\{y\} = 0$

$s\mathcal{L}y - 2 - 5\mathcal{L}y = 0$

$(s-5)\mathcal{L}\{y\} = 2$

$\mathcal{L}\{y\} = \frac{2}{(s-5)}$

$\mathrm{y} = \mathcal{L}^{-1}\{\frac{2}{(s-5)}\}$

$\mathrm{y} = 2\mathrm{e}^{5\mathrm{x}}$

2- $y^{'} + y = e^{x} \qquad y_{(0)} = 1$

$\mathcal{L}\{y^{'} + y\} = \mathcal{L}\{e^{x}\}$

$s\mathcal{L}\{y\} - y_{(0)} + \mathcal{L}\{y\} = \frac{1}{\mathrm{s}-1}$

$s\mathcal{L}\{y\} - 1 + \mathcal{L}\{y\} = \frac{1}{\mathrm{s}-1}$

$(s+1)\mathcal{L}\{y\} = \frac{1}{\mathrm{s}-1} + 1$

$(s+1)\mathcal{L}\{y\} = \frac{\mathrm{s}}{\mathrm{s}-1}$

$\mathcal{L}\{y\} = \frac{\mathrm{s}}{\mathrm{s}^2-1}$

$\mathrm{y} = \mathcal{L}^{-1}\{\frac{\mathrm{s}}{\mathrm{s}^2-1}\}$

$\mathrm{y} = \cosh \mathrm{x}$

3- $y^{''} + 2y^{'} + y = x \qquad y_{(0)} =- 3 \quad y^{'}_{(0)} =- 1$

$\mathcal{L}\{y^{''} + 2y^{'} + y\} = \mathcal{L}\{x\}$

$s^2\mathcal{L}\{y\} - sy_{(0)} - y^{'}_{(0)} + 2[s\mathcal{L}\{y\} - y_{(0)}] + \mathcal{L}\{y\} = \mathcal{L}\{x\}$

$s^2\mathcal{L}\{y\} + 3s + 1 + 2s\mathcal{L}\{y\} + 6 + \mathcal{L}\{y\} = \frac{1}{s^2}$

$(s^2 + 2s + 1)\mathcal{L}\{y\} = \frac{1}{s^2} - 3s - 7$

$(s^2 + 2s + 1)\mathcal{L}\{y\} = \frac{1-3s^3-7s^2}{s^2}$

$\mathcal{L}\{y\} = \frac{1-3s^3-7s^2}{s^2(s^2+2s+1)}$

$$\mathcal{L}\{y\} = \frac{1-3s^3-7s^2}{s^2(s^2+2s+1)} = \frac{As+B}{s^2} + \frac{Cs+D}{s^2+2s+1}$$

$$\mathcal{L}\{y\} = \frac{1-3s^3-7s^2}{s^2(s^2+2s+1)} = \frac{As^3+2As^2+As+Bs^2+2Bs+B+Cs^3+Ds^2}{s^2(s^2+2s+1)}$$

$$As^3 + 2As^2 + As + Bs^2 + 2Bs + B + Cs^3 + Ds^2 = 1 - 3s^3 - 7s^2$$

s^3: $A + C =- 3$ ....(1)

s^2: $2A + B + D =- 7$ ...(2)

s: $A + 2B = 0$ ....(3)

c: $B = 1 \;\rightarrow\; A =- 2 \;\rightarrow\; C =- 1 \;\rightarrow\; D =- 4$

$$\mathcal{L}\{y\} = \frac{1-3s^3-7s^2}{s^2(s^2+2s+1)} = \frac{-2s+1}{s^2} + \frac{-1s-4}{s^2+2s+1}$$

$$y = \mathcal{L}^{-1}\{\frac{-2s+1}{s^2} + \frac{-1s-4}{s^2+2s+1}\}$$

$$y = \mathcal{L}^{-1}\{\frac{-2s}{s^2} + \frac{1}{s^2} - \frac{s+1}{(s+1)^2} - \frac{3}{(s+1)^2}\}$$

$$y = \mathcal{L}^{-1}\{\frac{-2}{S} + \frac{1}{s^2} - \frac{1}{s+1} - \frac{3}{(s+1)^2}\}$$

$$y =- 2 + x - e^x - 3xe^{-x}$$

4- $y'' - 3y' + 4y = 0 \qquad y_{(0)} = 1 \quad y'_{(0)} = 5$

$$\mathcal{L}\{y'' - 3y' + 4y\} = \mathcal{L}\{0\}$$

$$s^2\mathcal{L}\{y\} - sy_{(0)} - y'_{(0)} - 3[s\mathcal{L}\{y\} - y_{(0)}] + 4\mathcal{L}\{y\} = 0$$

$$s^2\mathcal{L}\{y\} - s - 5 - 3s\mathcal{L}\{y\} + 3 + 4\mathcal{L}\{y\} = 0$$

$$(s^2 - 3s + 4)\mathcal{L}\{y\} = s + 2$$

$$\mathcal{L}\{y\} = \frac{s+2}{s^2-3s+4}$$

$$\mathcal{L}\{y\} = \frac{s+2}{s^2-3s+\frac{9}{4}-\frac{9}{4}+4}$$

$$\mathcal{L}\{y\} = \frac{s+2}{(s-\frac{3}{2})^2+\frac{7}{4}}$$

$$\mathcal{L}\{y\} = \frac{s+\frac{3}{2}-\frac{3}{2}+2}{(s-\frac{3}{2})^2+\frac{7}{4}}$$

$$\mathcal{L}\{y\} = \frac{s+\frac{3}{2}}{(s-\frac{3}{2})^2+\frac{7}{4}} - \frac{\frac{7}{2}}{(s-\frac{3}{2})^2+\frac{7}{4}}$$

$$y = \mathcal{L}^{-1}\{\frac{s+\frac{3}{2}}{(s-\frac{3}{2})^2+\frac{7}{4}} - \frac{\frac{7}{2}}{(s-\frac{3}{2})^2+\frac{7}{4}}\}$$

$$y = e^{\frac{3}{2}x}\cos\frac{\sqrt{7}}{2} + \sqrt{7}\,e^{\frac{3}{2}x}\sin\frac{\sqrt{7}}{2}$$

B) Using Laplace transformation to solve system of ODE

1- $y' + z = x \ldots\ldots(1) \qquad y_{(0)} = 1$

$z' + 4y = 0 \ldots.(2) \qquad z_{(0)} = 1$

$$\mathcal{L}\{y' + z\} = \mathcal{L}\{x\}$$
$$s\mathcal{L}\{y\} - y_{(0)} + \mathcal{L}\{z\} = \frac{1}{s^2}$$
$$s\mathcal{L}\{y\} - 1 + \mathcal{L}\{z\} = \frac{1}{s^2}$$
$$s\mathcal{L}\{y\} + \mathcal{L}\{z\} = \frac{1}{s^2} + 1$$
$$s\mathcal{L}\{y\} + \mathcal{L}\{z\} = \frac{1+s^2}{s^2} \ldots\ldots(1)$$

$$\mathcal{L}\{z' + 4y\} = \mathcal{L}\{0\}$$
$$s\mathcal{L}\{z\} - z_{(0)} + 4\mathcal{L}\{y\} = 0$$
$$s\mathcal{L}\{z\} - 1 + 4\mathcal{L}\{y\} = 0$$
$$s\mathcal{L}\{z\} + 4\mathcal{L}\{y\} = 1 \ldots\ldots.(2)$$

$$[s\mathcal{L}\{y\} + \mathcal{L}\{z\} = \frac{1+s^2}{s^2} \ldots\ldots..(1)](s)$$
$$[s\mathcal{L}\{z\} + 4\mathcal{L}\{y\} = 1 \ldots\ldots.(2)](-1)$$

$$s^2\mathcal{L}\{y\} + s\mathcal{L}\{z\} = \frac{1+s^2}{s} \ldots\ldots(1)$$
$$-s\mathcal{L}\{z\} - 4\mathcal{L}\{y\} = -1 \ldots\ldots(2)$$

$$s^2\mathcal{L}\{y\} - 4\mathcal{L}\{y\} = \frac{1+s^2}{s} - 1$$
$$(s^2 - 4)\mathcal{L}\{y\} = \frac{1-s+s^2}{s}$$
$$\mathcal{L}\{y\} = \frac{1-s+s^2}{s(s^2-4)}$$
$$\mathcal{L}\{y\} = \frac{1}{s(s^2-4)} - \frac{s}{s(s^2-4)} + \frac{s^2}{s(s^2-4)}$$
$$y = \mathcal{L}^{-1}\{\frac{1}{s(s^2-4)} - \frac{s}{s(s^2-4)} + \frac{s^2}{s(s^2-4)}\}$$
$$y = \frac{1}{2}\mathcal{L}^{-1}\{\frac{2}{s(s^2-4)}\} - \frac{1}{2}\mathcal{L}^{-1}\{\frac{2}{(s^2-4)}\} + \mathcal{L}^{-1}\{\frac{s}{(s^2-4)}\}$$

$$y = \frac{1}{2}\int_0^x \sinh 2x \; dx - \frac{1}{2}\sinh 2x + \cosh 2x$$
$$y = \frac{1}{4}[\cosh 2x]_0^x - \frac{1}{2}\sinh 2x + \cosh 2x$$
$$y = \frac{1}{4}\cosh 2x - \frac{1}{4}\cosh 0 - \frac{1}{2}\sinh 2x + \cosh 2x$$
$$y = \frac{5}{4}\cosh 2x - \frac{1}{2}\sinh 2x - \frac{1}{4}$$

From (1) $z = x - y'$

$$z = x - \frac{5}{2}\sinh 2x - \cosh 2x$$

2- $y'' + z + y = 0 \ldots\ldots (1)$ $\qquad y_{(0)} = 0 \;\; y'_{(0)} = 0$

$z' + y' = 0 \ldots\ldots\ldots\ldots (2)$ $\qquad z_{(0)} = 1$

$$\mathcal{L}\{y'' + z + y\} = \mathcal{L}\{0\}$$
$$s^2\mathcal{L}\{y\} - sy_{(0)} - y'_{(0)} + \mathcal{L}\{z\} + \mathcal{L}\{y\} = 0$$
$$(s^2 + 1)\mathcal{L}\{y\} + \mathcal{L}\{z\} = 0 \ldots\ldots (1)$$

$$\mathcal{L}\{z' + y'\} = \mathcal{L}\{0\}$$
$$s\mathcal{L}\{y\} - y_{(0)} + s\mathcal{L}\{z\} - z_{(0)} = 0$$
$$s\mathcal{L}\{y\} + s\mathcal{L}\{z\} = 1 \ldots\ldots\ldots\ldots (2)$$

$$[(s^2 + 1)\mathcal{L}\{y\} + \mathcal{L}\{z\} = 0 \ldots\ldots (1)](-1)$$
$$[s\mathcal{L}\{y\} + s\mathcal{L}\{z\} = 1 \ldots\ldots\ldots\ldots (2)] \div (s)$$

$$-(s^2 + 1)\mathcal{L}\{y\} - \mathcal{L}\{z\} = 0 \ldots\ldots (1)$$
$$\mathcal{L}\{y\} + \mathcal{L}\{z\} = \frac{1}{s} \ldots\ldots\ldots\ldots\ldots\ldots (2)$$

$$-(s^2 + 1)\mathcal{L}\{y\} + \mathcal{L}\{y\} = \frac{1}{s}$$
$$-s^2\mathcal{L}\{y\} = \frac{1}{s} \quad \rightarrow \quad \mathcal{L}\{y\} = \frac{-1}{s^3}$$
$$y = \mathcal{L}^{-1}\{\frac{-1}{s^3}\} \;\rightarrow\; y = -\frac{1}{2}\mathcal{L}^{-1}\{\frac{2}{s^3}\} \;\rightarrow\; y = -\frac{1}{2}x^2$$

From (1) $y'' + z + y = 0$

$$z = \frac{1}{2}x^2 + 1$$

3- $z'' + y' = \cos x \; \ldots\ldots (1) \qquad z_{(0)} = -1 \;\; z'_{(0)} = -1$

$y'' - z = \sin x \; \ldots\ldots (2) \qquad y_{(0)} = 1 \;\; y'_{(0)} = 0$

$\mathcal{L}\{z'' + y'\} = \mathcal{L}\{\cos x\}$

$s^2\mathcal{L}\{z\} - sz_{(0)} - z'_{(0)} + s\mathcal{L}\{y\} - y_{(0)} = \frac{s}{s^2+1}$

$s^2\mathcal{L}\{z\} + s + 1 + s\mathcal{L}\{y\} - 1 = \frac{s}{s^2+1}$

$s^2\mathcal{L}\{z\} + s\mathcal{L}\{y\} = \frac{s}{s^2+1} - s$

$s^2\mathcal{L}\{z\} + s\mathcal{L}\{y\} = \frac{-s^3}{s^2+1} \ldots\ldots (1)$

$\mathcal{L}\{y'' - z\} = \mathcal{L}\{\sin x\}$

$s^2\mathcal{L}\{y\} - sy_{(0)} - y'_{(0)} - \mathcal{L}\{z\} = \frac{1}{s^2+1}$

$s^2\mathcal{L}\{y\} - s - \mathcal{L}\{z\} = \frac{1}{s^2+1}$

$s^2\mathcal{L}\{y\} - \mathcal{L}\{z\} = \frac{1}{s^2+1} + s$

$s^2\mathcal{L}\{y\} - \mathcal{L}\{z\} = \frac{1+s^3+s}{s^2+1} \ldots\ldots (2)$

$s^2\mathcal{L}\{z\} + s\mathcal{L}\{y\} = \frac{-s^3}{s^2+1} \ldots\ldots (1)$

$[s^2\mathcal{L}\{y\} - \mathcal{L}\{z\} = \frac{1+s^3+s}{s^2+1} \ldots\ldots (2)](s^2)$

$s^2\mathcal{L}\{z\} + s\mathcal{L}\{y\} = \frac{-s^3}{s^2+1} \ldots\ldots (1)$

$s^4\mathcal{L}\{y\} - s^2\mathcal{L}\{z\} = \frac{s^2+s^5+s^3}{s^2+1} \ldots\ldots (2)$

$s^4\mathcal{L}\{y\} + s\mathcal{L}\{y\} = \frac{s^2+s^5}{s^2+1} \;\rightarrow\; (s^4 + s)\mathcal{L}\{y\} = \frac{s^2+s^5}{s^2+1}$

$\mathcal{L}\{y\} = \frac{s^2(s^3+1)}{s(s^3+1)(s^2+1)} \;\rightarrow \mathcal{L}\{y\} = \frac{s}{s^2+1}$

$\mathrm{y} = \mathcal{L}^{-1}\{\frac{s}{s^2+1}\} \;\rightarrow\; \mathrm{y} = \cos \mathrm{x}$

From (2) $y'' - z = \sin x$

$\mathrm{z} = -\cos \mathrm{x} - \sin \mathrm{x}$

H.W. Find the general solution of the differential equation

1- $y'' + 2y' + y = 2xe^{-x}$ $\quad y_{(0)} = 4 \quad y'_{(0)} = 2$

2- $y'' + y = 6\sin 2x$ $\quad y_{(0)} = -1 \quad y'_{(0)} = -4$

3- $y'' + 3y' + 2y = 4x^2$ $\quad y_{(0)} = 0 \quad y'_{(0)} = 0$

4- $y'' - 2x = 2 \ldots\ldots\ldots (1)$ $\quad y_{(0)} = 2 \quad y'_{(0)} = 2$

$y + x' = 5e^x + 1 \ldots (2)$ $\quad x_{(0)} = 1$

5- $x - y = 2 \ldots. (1)$

$x'' - y' = e^x \ldots. (2)$ $\quad x_{(0)} = 0 \quad x'_{(0)} = 1 \quad y_{(0)} = 0$