Double integral التكامل المضاعف

The properties of double integrals:

Double integrals over region R of continuous functions have the following properties:

1- $\iint_R k(f(x,y))dx\,dy = k \iint_R (f(x,y))dx\,dy$

نلاحظ هنا *f(x,y)* هي دالة معرفة بمتغيرين هما *x , y*

2- $\iint_R [f(x,y)) \pm g(x,y))]\,dx\,dy = \iint_R (f(x,y))dxdy \pm \iint_R (g(x,y))dx\,dy$

3- *If* $R = R_1 \cup R_2$ *then*

$$\iint_R (f(x,y))dx = \iint_{R_1} f(x,y)dxdy + \iint_{R_2} f(x,y)dx\,dy$$

Theorem: (Fubini's)(first order)

If $f(x,y)$ *is continuous function on the rectangular reign*

$$R: a \le x \le b\,,\, c \le y \le d \text{ then}$$

$$\iint_R f(x,y)dx\,dy = \int_c^d \int_a^b f(x,y)dxdy = \int_a^b \int_c^d f(x,y)dydx$$

البداية مع x يعني نكامل بالنسبة
لل x وال y يعتبر ثابت

Example:

Calculate $\int_2^3 \int_0^1 (x + 3y)\, dx\, dy$

Solution:

$$\int_2^3 [[\frac{x^2}{2} + 3\, y\, x]_0^1] dy$$

$$= \int_2^3 [(\frac{(1)^2}{2} + 3\, y\, (1)) - (0)] dy$$

$$= \int_2^3 (\frac{1}{2} + 3\, y)\, dy$$

$$= [(\frac{1}{2} y + 3\, \frac{y^2}{2})]_2^3$$

$$= \left(\frac{1}{2}(3) + 3\, \frac{(3)^2}{2}\right) - (\frac{1}{2}(2) + 3\, \frac{(2)^2}{2})$$

$$= (\frac{30}{2}) - (7)$$

$$= 8$$

Example:

Calculate $\int_0^1 \int_2^3 (x + 3y)\, dy\, dx$

Solution:

$$\int_0^1 \int_2^3 (x + 3y)\, dy\, dx = \int_0^1 [[xy + 3\frac{y^2}{2}]_2^3]\, dx$$

$$\int_0^1 (x(3) + 3\frac{(3)^2}{2}) - (x(2) + 3\frac{(2)^2}{2})\, dx$$

$$\int_0^1 (x + \frac{15}{2})\, dx$$

$$= [\frac{x^2}{2} + \frac{15}{2}x]_0^1$$

$$= (\frac{(1)^2}{2} + \frac{15}{2}(1)) - (0)$$

$$= \frac{1}{2} + \frac{15}{2}$$

$$= 8$$

$$\int_1^2 \left(x^4 + 3\,x^2 - x + 1\right) dx = [\frac{x^5}{5} + \frac{3x^3}{3} - \frac{x^2}{2} + x]_1^2$$

$$=[\frac{(2)^5}{5} + \frac{3(2)^3}{3} - \frac{(2)^2}{2} + 2]-[\frac{(1)^5}{5} + \frac{3(1)^3}{3} - \frac{(1)^2}{2} + 1]$$

$$= [\frac{32}{5} + 8 - 2 + 2] - [\frac{1}{5} + 1 - 1 + 1]$$

$$= \frac{31}{5} + 7$$

$$= \frac{66}{5}$$

Example:

Evaluate the integral

$$\int_0^{\frac{\pi}{2}} \cos x \quad dx$$

Solution:

$$\int_0^{\frac{\pi}{2}} \cos x \quad dx = [\sin x]_0^{\frac{\pi}{2}}$$

$$= \sin\frac{\pi}{2} - \sin 0$$

$$= 1$$

Properties of definite integral **خواص التكامل المحدد**

1- If a is in the domain of f we have

$$\int_a^a f(x)dx = 0$$

2- $\int_a^b f(x)dx = -\int_b^a f(x)dx$

3- $\int_a^b kf(x)dx = k\int_a^b f(x)dx$

4- $\int_a^b (f(x) \pm g(x))dx = \int_a^b f(x)dx \pm \int_a^b g(x)dx$

5- If $f(x)$ is continuous on [a, c] such that $a < b < c$ then

$$\int_a^b f(x)dx + \int_b^c f(x)dx = \int_a^c f(x)dx$$

Example:

Evaluate the following integrals

1- $\int_0^3 (x^2 + 2x)dx$

$$\int_0^3 (x^2 + 2x)dx \ = [\frac{x^3}{3} + 2\frac{x^2}{2}]_0^3$$

$$= [(\frac{(3)^3}{3} + 2\frac{(3)^2}{2}) - (\frac{(0)^3}{3} + 2\frac{(0)^2}{2})]$$

$$= 18$$

2- $\int_{-1}^0 (x + 1)dx$

$$\int_{-1}^0 (x + 1)dx = [\frac{x^2}{2} + x]_{-1}^0$$

$$= (0) - (\frac{1}{2} - 1)$$

$$= \frac{1}{2}$$

3- $\int_4^5 \frac{dx}{(x-3)}$

$$\int_4^5 \frac{dx}{(x-3)} = [ln\ (x-3)]_4^5$$

$$= ln\ (5-3) - ln\ (4-3)$$

$$= ln\,(2) - ln\,(1)$$

$$= ln\,(2)$$

4- Let $f(x) = f(x) = \begin{cases} 3x - 2, & x < 2 \\ x^2, & x \geq 2 \end{cases}$ find $\int_0^6 f(x)dx$

Solution:

$$\int_0^6 f(x)dx = \int_0^2 (3x - 2)dx + \int_2^6 (x^2) f(x)dx$$

$$= [\frac{3x^2}{2} - 2x]_0^2 + [\frac{x^3}{3}]_2^6$$

$$(\frac{3(2)^2}{2} - 4) + [(\frac{(6)^3}{3}) - (\frac{(6)^2}{3})]$$

$$=$$

Application of the Definite Integral ***تطبيقات التكامل المحدد***

The area under curve ***المساحة تحت المنحني***

We can find the area between the graph of $y = f(x)$ and the x-axis over the interval [a,b] by the following formula

$$\text{Area}= \left|\int_a^b f(x)dx\right|$$

م / تم وضع علامة المطلق حول التكامل لان المساحة موجبة دائما.

<u>Example:</u>

Find the Area between the curve $y = x^2 - 2x + 3$ and the x-axis from $x = 0 \; to \;\; x = 3$

<u>solution:</u>

$$\text{Area}= \left|\int_a^b f(x)dx\right|$$

$$=\left|\int_0^3 (x^2 - 2x + 3)dx\right|$$

$$= \left|[\frac{x^3}{3} - \frac{2x^2}{2} + 3x]_0^3\right|$$

$$= \left|(\frac{(3)^3}{3} - \frac{2(3)^2}{2} + 3(2)) - (\frac{(0)^3}{3} - \frac{2(0)^2}{2} + 3(0))\right|$$

$$= \left|(\frac{27}{3} - 9 + 9) - (0)\right|$$

=9

Example:

Find the Area between the curve $y = 4 - x^2$ and the x-axis.

م ا في هذا المثال اعطى منحني الدالة فقط ولم يعطي حدود التكامل وعليه يجب ايجاد حدود التكامل اولا.

ولايجاد حدود التكامل نساوي الدالة بالصفر ونجد قيم x التي تمثل حدود التكامل.

Solution:

$$y = 4 - x^2$$

$$\Rightarrow 4 - x^2 = 0$$

$$\Rightarrow (2 - x)(2 + x) = 0$$

$$\Rightarrow either\ x = -2 \quad or \quad x = 2$$

الان اصبحت حدود التكامل a=-2 ***و*** b=2

$$\text{Area}= \left|\int_a^b f(x)dx\right|$$

$$= \left|\int_{-2}^{2}(4 - x^2)\right|$$

$$= \left|[4x - \frac{x^3}{3}]_{-2}^{2}\right|$$

$$= \left|(4(2) - \frac{(2)^3}{3}) - (4(-2) - \frac{(-2)^3}{3})\right|$$

$$= \frac{32}{3}$$

Example:

Find the total Area bounded by the curve $y = x^3 - 4x$ and the x-axis.

Solution:

$$x^3 - 4x = 0 \Rightarrow x(x^2 - 4) = 0$$

$$x(x - 2)(x + 2) = 0$$

either $x = 0$ or $x = 2$, $x = -2$

نلاحظ هنا اصبحت لدينا ثلاثة حدود

∴ Area=$|A_1| + |A_2|$

$$= \left|\int_{-2}^{0}(x^3 - 4x)dx\right| + \left|\int_{0}^{2}(x^3 - 4x)dx\right|$$

$$|A_1| = \left|\int_{-2}^{0}(x^3 - 4x)dx\right| = [\frac{x^4}{4} - \frac{4x^2}{2}]_{-2}^{0} = \left|0 - (\frac{16}{4} - 8)\right|$$

$$= 4$$

$$|A_2| = \left|\int_{0}^{2}(x^3 - 4x)dx\right| = [\frac{x^4}{4} - \frac{4x^2}{2}]_{0}^{2} = \left|[\frac{16}{4} - 8] - [0]\right|$$

$$= 4$$

Thus Area=$|A_1| + |A_2| = 4 + 4 = 8$

Example:

Find the total Area of the region between the curves

$y = x^3 - x^2 - 6x$ and the x-axis over the interval [-2,2].

Solution:

$$x^3 - x^2 - 6x = 0 \Rightarrow x(x^2 - x - 6) = 0$$

$$\Longrightarrow \boldsymbol{x(x-3)(x+2) = 0}$$

Either $x = 0$ or $x = 3$, $x = -2$

نلاحظ هناك ثلاث قيم لل

نحدد القيم التي داخل الفترة المعطاة في السؤال فقط

وعليه -2, 0 هي داخل الفترة ولكن 3 خارج الفترة وعليه تهمل في الحل

Area= $|A_1| + |A_2|$

$$\text{Area}= \left|\int_{-2}^{0} f(x)dx\right| + \text{Area}= \left|\int_{0}^{2} f(x)dx\right|$$

$$|A_1| = \left|\int_{-2}^{0} (x^3 - x^2 - 6x)\right| = \left|[\frac{x^4}{4} - \frac{x^3}{3} - \frac{6x^2}{2}]_{-2}^{0}\right|$$

$$= \left|[0] - [\frac{16}{4} - \frac{-8}{3} - \frac{24}{2}]\right| = \frac{16}{3}$$

$$|A_2| = \left|\int_{0}^{2} (x^3 - x^2 - 6x)\right| = \left|[\frac{x^4}{4} - \frac{x^3}{3} - \frac{6x^2}{2}]_{0}^{2}\right|$$

$$= \left|[\frac{16}{4} - \frac{8}{3} - \frac{24}{2}] - [0]\right|$$

$$\frac{32}{3}$$

Thus Area= $|A_1| + |A_2| = \frac{16}{3} + \frac{32}{3} = 16$