Foundations of Mathematics

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Chapter Four
Mappings

Lecture 12.:Mapping: Definitions, Representations and Types

Introduction

Mappings (Functions) are one of the most fundamental concepts in mathematics. They serve as a bridge between different mathematical structures and have widespread applications in science, engineering, and everyday life. A mapping (function) defines a relationship between two sets, where each input is associated with exactly one output. In this lecture, we will explore the definition, types, and properties of mappings (functions), along with their significance in mathematics.

Mapping:

A relation f from a set A (domain of f) to B (codomain of f) is called a mapping or function if assigns each element in A to exactly one element of B. mathematically, a relation f denoted as $f: A \to B$ is mapping or function if it satisfies the following conditions:

- Closure $\forall \mathbf{x} \in \mathbf{A} \implies \mathbf{f(x)} \in \mathbf{B}$
- Well-defined $\forall \mathbf{x_1, x_2} \in \mathbf{A},\ \mathbf{If\ x_1 = x_2\ then\ f(x_1) = f(x_2)}$

Representation of Mapping (Functions):

A mappings (Functions) can be represented in different ways:

- Algebraically: Using an equation (e.g. $f(x) = x + 3$).
- Graphically: Using a Cartesian coordinate system.
- Tabular Form: Listing input-output pairs.
- Verbal Description: Describing the relationship in words.

Examples: 4.1. 1) let $f: Z \to Z$ be a relation such that $f(x) = x + 3$. Is f a mapping?

Solution: Closure let $x \in Z \Rightarrow f(x) = x + 3 \in Z$ then f is Closure

Well-defined let $x_1, x_2 \in Z$, such that $x_1 = x_2$

Since $x_1 = x_2 \Rightarrow x_1 + 3 = x_2 + 3 \Rightarrow f(x_1) = f(x_2)$ then f is Well-defined

f is mapping.

2) let $f: Z \to R$ be a relation such that $f(x) = \sqrt{x^2 + 1}$. Is f a mapping?

Solution: Closure let $x \in Z \Rightarrow f(x) = \sqrt{x^2 + 1}. \in R$ then f is Closure

Well-defined let $x_1, x_2 \in Z$, such that $x_1 = x_2$

Since $x_1 = x_2 \Rightarrow x_1^2 = x_2^2 \Rightarrow x_1^2 + 1 = x_2^2 + 1 \Rightarrow \sqrt{x_1^2 + 1} = \sqrt{x_2^2 + 1} \Rightarrow f(x_1) = f(x_2)$ then f is Well-defined

f is mapping.

3) let $f: Z \to Z$ be a relation such that $f(x) = \sqrt{x^2 + 1}$. Is f a mapping?

Solution: Since $1 \in Z \Rightarrow f(1) = \sqrt{1^2 + 1} = \sqrt{2} \notin Z$ then f is not closure

f is not mapping.

4) let $f: R \to R$ be a relation such that $f(x) = \begin{cases} x & ; x \geq 1 \\ -x & ; x \leq 1 \end{cases}$ Is f a mapping?

Solution: Since $1 \in Z \Rightarrow f(1) = 1 \; and \; f(1) = -1$ then f is not Well-defined

f is not mapping.

Definition (4.1): Let $f: A \to B$ be a mapping . We define the following sets

1) Domain of f denoted by D_f the set of all possible input values (A).

2) Codomain of f denoted by Cod_f the set (B).

3) Range of f denoted by R_f the set of all possible output values$R_f \subseteq B$.

4) If $f(x) = y$ then y is called the image of x and x is called the preimage of y.

Examples: 4.2. Find the domain and the range of the following functions:

1. $f: N \to N; \quad f(x) = x + 1$
2. $f: Z \to Z; \quad f(x) = x^2$
3. $f: R \to R; \quad f(x) = x^3 - 1$
4. $f: R \to R; \quad f(x) = |x^2 - 1| - 3$
5. $f: R \to R; \quad f(x) = \frac{x}{2x^2+1}$

Solution: 1. $D_f = N \quad , \ R_f = \{2,3, \dots\}$

2. $D_f = Z \quad , \ R_f = \{1,4,9,16, \dots\}$

3. $D_f = R \quad , \ R_f = R$

4. $D_f = R \quad , \ R_f = [-3, \infty)$

5. $D_f = R \quad , \ R_f = R - \{0\}$

Types of Functions:

Mappings (Functions) can be classified based on their properties:

1. **Constant Mapping (Function)**: A mapping (function) where the output remains the same for all inputs foe example $f(x) = 5$

2. **Identity Mapping (Function)**: A mapping (function) where each element of the domain to itself. Mathematically, it is defined as:

$$I_A: A \to B, I_A(x) = x$$

3. **Injective Mapping (Function)**: A mapping (function) $f: A \to B$ is called injective or (1-1) if different inputs produce different outputs. Mathematically,

f is (1-1) If $\forall x_1, x_2 \in$ A and $f(x_1) = f(x_2)$ then $x_1 = x_2$

f is not (1-1) If $\exists x_1, x_2 \in$ A and $f(x_1) = f(x_2)$ then $x_1 \neq x_2$

4. **Surjective Mapping (Function)**: A mapping (function) $f: A \to B$ is called surjective or (onto) if every element in B has at least one preimage in A. Mathematically,

f is (onto) $\Leftrightarrow Cod_f = R_f$

5. **Bijective Mapping (Function)**: A mapping (function) $f: A \to B$ is called bijective if and only if f is 1-1 and onto.

Examples: 4.3. Discuss the following functions in terms of their types:

1) $f: N \to N;\ f(x) = 5$

f is constant

f is not identity

f is not (1-1) since $f(1) = f(2)\ but\ 1 \neq 2$

f is not onto since $Cod_f \neq R_f$

f is not bijective

2) $f: N \to N;\ f(x) = x$

f is not constant

f is identity since $\forall x \in N\ ; f(x) = x$

f is (1-1) since if $f(x_1) = f(x_2)\ then\ x_1 = x_2$

f is onto since $Cod_f = R_f = N$

f is bijective

3) $f: R \to R;\ f(x) = |x|$

f is not constant

f is not identity since $f(-1) = 1$

f is not (1-1) since if $f(-1) = f(1) = 1\ but - 1 \neq 1$

f is not onto since $Cod_f \neq R_f$

f is not bijective

Examples: 4.4. Which of the following functions are injective? Surjective?

Bijective?

1) $f: R \to R;\ \ f(x) = 2x^2 - 3$

2) $f: R \to R;\ \ f(x) = \begin{cases} x^2, & x \geq 0 \\ -x^2, & x < 0 \end{cases}$

Solution: 1) f is (1-1) since if $f(x_1) = f(x_2)$ then

Either $f(x_1) = f(x_2)$ are positive $\Rightarrow x_1^2 = x_2^2$ *and* x_1, x_2 are positive $\Rightarrow x_1 = x_2$

Or $f(x_1) = f(x_2)$ are negative $\Rightarrow -x_1^2 = -x_2^2$ *and* x_1, x_2 are negative $\Rightarrow x_1 = x_2$

f is onto since

For each $y \in R \Rightarrow x = \sqrt{y} \in R \Rightarrow y = x^2 = f(x)$

f is bijective since f is (1-1) and onto.

2) H.W.

Conclusion

Mappings (Functions) are a fundamental concept in mathematics that describe relationships between variables. They appear in nearly every branch of mathematics and have practical applications in science, engineering, and technology. Understanding their properties and classifications helps in solving mathematical problems efficiently.

Exercises

Exercise 1: Are the following relation mappings

1) $f: N \to N\ ;\ f(x) = \frac{x}{|x|-5}$

2) $f: R \to R\ ;\ f(x) = \frac{1}{\sqrt{x-5}}$

Exercise 2: Which of the following functions are injective? Surjective?

Bijective?

1) $f: R \to R;\ \ f(x) = \sqrt{x^2 - 9}$

2) $f: R \to R;\ \ f(x) = \begin{cases} 2x^2 - 1\,, & x \geq 1 \\ 1 \quad\ \ , & x < 1 \end{cases}$

3) $f: R \to [1, \infty);\ \ f(x) = |x - 4| + 1$

Exercise 3: Give an example of a mapping such that (1-1) and not onto?

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