Foundations of Mathematics

Instructor:
Uday Jabbar Quaez and Ruqaya Ibrahim
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Department of Mathematics, College of Education, Al-Mustansiriyah University

Chapter Five
Cardinality of Sets

Lecture 15.: Equivalent of Sets

Introduction

In mathematics, the concept of sets is a fundamental building block for many mathematical branches. Among the associated concepts are set equivalence and the distinction between finite and infinite sets. These concepts help in understanding complex mathematical structures and dealing with infinite sets.

Equivalent of Two Sets:

Two sets A and B are considered equivalent (i.e., they have the same number of elements) if there exists a bijective function between them a function that is both one-to-one and onto.

Symbolically:

$A \blacksquare B \iff \exists f: A \to B \text{ is bijective}$

$A \not\blacksquare B \iff \forall f: A \to B \text{ is not bijective}$

Examples: 5.1.

1) Let $A = \{1,2,3\}$ and $B = \{a, b, c\}$. Then $A \blacksquare B$ since

If we choose $f = \{(1, a), (2, b), (3, c)\}$, f is bijective $((1-1)$ and onto $)$

2)) Let $A = \{1,2,3\}$ and $B = \{a, b\}$. Then $A \not\blacksquare B$ since

If we assume that there exists a function $f: A \to B$ is bijective this means that f is $(1-1)$, precisely $f(1) \neq f(2) \neq f(3)$, therefore the range has

three elements but the codomain has two elements this leads contradiction. Then, $\forall f: A \to B \;\; is\ not\ bijective$ and $A \not\approx B$.

3) $N \blacksquare N$ since

If we choose $I_N: N \to N,\ I_N(x) = x$ then I_N is bijective (check)

Therefore, $N \blacksquare N$

4) Let $A = N \;\; and\ B = \{2,4,6, \dots\}$. Then, $A \blacksquare B$ since

Define $f: A \to B; f(x) = 2x$, then f is bijective. (check)

Consequently, $A \blacksquare B$

Examples: 5.2.

1. Show that $N \approx Z^-$.

Solution: Define $f: N \to Z^-; f(x) = -x$, To prove

a. f is mapping , let $x \in N \Longrightarrow -x \in Z^- \Longrightarrow f(x) \in Z^- \;\; then\ f\ is\ closure$
Let $x_1, x_2 \in N;\ x_1 = x_2 \Longrightarrow -x_1 = -x_2 \Longrightarrow f(x_1) = f(x_2)$ then f is well defined therefore, f is mapping.

b. f is $(1-1)$. Let $f(x_1) = f(x_2) \Longrightarrow -x_1 = -x_2 \Longrightarrow x_1 = x_2$ then f is$(1-1)$

c. f is $onto$. Let $y \in Z^- \Longrightarrow x = -y \in N \Longrightarrow -x = y \Longrightarrow y = f(x)$ then f is onto. Therefore, f is bijective and $N \approx Z^-$

2. $N \approx B = \{3,6,9, \dots\}$ (H.W.)

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Theorem (5.1): An equivalent relation $(\approx)$ on a set is an equivalence relation.

Proof: An equivalence relation on a set must satisfy three properties: reflexivity, symmetry, and transitivity.

1. Reflexivity: to prove $A \approx A$ for each set A
   Define $I_A: A \to A,\ I_A(x) = x$ then I_A is bijective this means $A \approx A$

2. Symmetry: Let A and B are two sets such that $A \approx B$. Then, to prove $B \approx A$. $A \approx B \Longrightarrow \exists f: A \to B\ is\ bijective \Longrightarrow \exists f^{-1}: B \to A\ is\ bijective \Longrightarrow B \approx A$.

3. Transitivity: Let A,B and C are sets such that $A \approx B$ and $B \approx C$. Then, to prove $A \approx C$.
   $A \approx B\ \exists f: A \to B\ is\ bijective$
   $B \approx C\ \exists g: B \to C\ is\ bijective$
   $\therefore \exists gof: A \to C$ is $bijective \Longrightarrow A \approx C$
   $\therefore \approx$ is transitive
   $\therefore \approx$ is an equivalence relation

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Examples: 5.3. Show that $N \approx Z$.

Solution: Define $f: N \to Z; f(x) = \begin{cases} \frac{x}{2} & ; x\ is\ even \\ -\frac{x-1}{2} & ; x\ is\ odd \end{cases}$

To prove

1. f is mapping , let $x \in N \Longrightarrow if\ x\ is\ even\ then\ \frac{x}{2} \in Z\ or\ x\ is\ odd\ then\ -\frac{x-1}{2} \Longrightarrow f(x) \in Z\ \ then\ f\ is\ closure$

   Let $x_1, x_2 \in N;\ x_1 = x_2 \Longrightarrow if\ \ x_1, x_2\ are\ even \Longrightarrow \frac{x_1}{2} = \frac{x_2}{2} \Longrightarrow$
   $f(x_1) = f(x_2)$ or $x_1, x_2\ are\ odd \Longrightarrow -\frac{x_1-1}{2} = -\frac{x_2-1}{2} \Longrightarrow$
   $f(x_1) = f(x_2)$ then f is well defined therefore, f is mapping.
2. f is $(1-1)$. Let $f(x_1) = f(x_2) \Longrightarrow$
   $if\ f(x_1)\ and\ f(x_2)\ are\ positive \Longrightarrow \frac{x_1}{2} = \frac{x_2}{2} \Longrightarrow x_1 = x_2$ or
   $f(x_1)\ and\ f(x_2)\ are\ negative \Longrightarrow -\frac{x_1-1}{2} = -\frac{x_2-1}{2} \Longrightarrow x_1 = x_2$
   then f is$(1-1)$
3. f is $onto$. Let $y \in Z \Longrightarrow if\ y\ is\ \ positive\ \ x = 2y \in N \Longrightarrow\ \ y = f(x)$

$$\text{Or } y \; is \;\; negative \;\; x = -2y + 1 \in N \implies y = f(x) \;\; \text{then } f \text{ is onto}$$

Then $f \; is \; bijective \implies N \approx Z$

Finite and Infinite Sets:

A set A is said to be finite if it is empty or if it contains exactly m elements otherwise A *is* infinite**.**

Remark (5.2):

- The elements number of a finite set A is called the size of A and denoted by $n(A)$ *or* $\#(A)$ *or* $|A|$.
- If A is an infinite set then its size $(n(A) \; or \; \#(A) \; or \; |A|)|A|$ is not defined (does not exist)

Examples: 5.4.

1. $A = \{1,2,3\}, then \; A \; is \; finite \; set \; and \;\; n(A) = 3$
2. $A = \emptyset \;, then \; A \; is \; finite \; set \; and \;\; \#(A) = 0$
3. $A = \{\emptyset\}, then \; A \; is \; finite \; set \; and \;\; \#(A) = 1$
4. $A = \{1, \emptyset, \{1,3\}, [0,1], N\} then \; A \; is \; finite \; set \; and \; |A| = 5$
5. $A = Z, then \; A \; is \; infinite \; set \; and \;\; \#(A) \; is \; not \; defined$

Exercises

Exercise 1: Let $A = (9,40]$ be given. Is A finite or infinite? Find $n(A)$?

Exercise 2: $Let \; A = \{x, y, z\}, \;\; B = \{0, -1\}. \; Is \; A \approx B? \; Is \; B \approx A?$

Exercise 3: $Show \; that \; N \approx B = \{5,10,15,20, \dots\}$

References

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