

Matrices

Prerequisites: Adding, subtracting, multiplying and dividing numbers; elementary row operations.

Maths Applications: Solving systems of equations; describing geometric transformations; deriving addition formulae.

Real-World Applications: Balancing chemical equations; flight stopover information; currents in electrical circuits; formulation of fundamental physical laws.

Basic Definitions

Definition:

A **matrix** is a rectangular array of numbers (aka **entries** or **elements**) in parentheses, each entry being in a particular **row** and **column**.

Definition:

The **order** of a matrix is given as $m \times n$ (read **m by n**), where m is the number of rows and n the number of columns and is written as,

$$A \equiv (a_{ij})_{m \times n} \stackrel{\text{def}}{=} \begin{pmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{pmatrix}$$

The element in row i and column j of a matrix is written as a_{ij} and called the **(i, j)th entry of A**.

In this course, we will deal almost exclusively with matrices that have orders 2×2 and 3×3 .

Definition:

The **main diagonal** (aka **leading diagonal**) of any matrix is the set of entries a_{ij} where $i = j$.

Special cases arise when either $m = 1$ or $n = 1$.

Definition:

A **row matrix** is a $1 \times n$ matrix and is written as,

$$(a_{11} \ a_{12} \ \dots \ a_{1(n-1)} \ a_{1n})$$

A **column matrix** is a $m \times 1$ matrix and is written as,

$$\begin{pmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{(m-1)1} \\ a_{m1} \end{pmatrix}$$

The case when $m = n$ is a very important one.

Definition:

A **square matrix (of order $m \times m$)** is a matrix with the same number of rows as columns (equal to m) and is written as,

$$\begin{pmatrix} a_{11} & a_{12} & \dots & a_{1m} \\ a_{21} & a_{22} & \dots & a_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mm} \end{pmatrix}$$

Definition:

The **identity matrix (of order m)** is the $m \times m$ matrix all of whose entries are 0 apart from those on the main diagonal, where they all equal 1,

$$I_m \stackrel{\text{def}}{=} \begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \end{pmatrix}$$

Definition:

The **zero matrix (of order $m \times n$)** is the $m \times n$ matrix all of whose entries are 0,

$$O_{m \times n} \stackrel{\text{def}}{=} \begin{pmatrix} 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 0 \end{pmatrix}$$

For a square zero matrix, sometimes the notation O_m is used.

Matrix Algebra*Addition, Subtraction and Scalar Multiplication*Definition:

The **matrix sum of A and B** is obtained by adding corresponding entries of A and B ,

$$(a + b)_{ij} \stackrel{\text{def}}{=} a_{ij} + b_{ij}$$

The **matrix difference of A and B** is obtained by subtracting the entries of B from the corresponding ones in A ,

$$(a - b)_{ij} \stackrel{\text{def}}{=} a_{ij} - b_{ij}$$

Note that matrix addition and subtraction only makes sense if A and B both have the same order.

Example 1

Add the matrices $A = \begin{pmatrix} 0 & -3 \\ 5 & 2 \end{pmatrix}$ and $B = \begin{pmatrix} 13 & 1 \\ -4 & 5 \end{pmatrix}$.

As the matrices have the same order, they can be added.

$$\begin{aligned} A + B &= \begin{pmatrix} 0 & -3 \\ 5 & 2 \end{pmatrix} + \begin{pmatrix} 13 & 1 \\ -4 & 5 \end{pmatrix} \\ &= \begin{pmatrix} 0 + 13 & -3 + 1 \\ 5 + (-4) & 2 + 5 \end{pmatrix} \\ &= \begin{pmatrix} 13 & -2 \\ 1 & 7 \end{pmatrix} \end{aligned}$$

Example 2

Find the difference $P - Q$ where $P = \begin{pmatrix} 4 & -2 & 0 \\ 1 & 0 & 37 \end{pmatrix}$ and

$$Q = \begin{pmatrix} -4 & -2 & 2 \\ -8 & 3 & 6 \end{pmatrix}.$$

$$\begin{aligned} P - Q &= \begin{pmatrix} 4 & -2 & 0 \\ 1 & 0 & 37 \end{pmatrix} - \begin{pmatrix} -4 & -2 & 2 \\ -8 & 3 & 6 \end{pmatrix} \\ &= \begin{pmatrix} 8 & 0 & -2 \\ 9 & -3 & 31 \end{pmatrix} \end{aligned}$$

Definition:

The **scalar multiplication** of A by k ($k \in \mathbb{R}$) is obtained by multiplying each entry of A by k ,

$$(ka)_{ij} \stackrel{\text{def}}{=} ka_{ij}$$

Example 3

If $A = \begin{pmatrix} 4 & -4 & 0 \\ 5 & 6 & 1 \\ 8 & 9 & -1 \end{pmatrix}$, calculate $\frac{1}{2}A$.

$$\begin{aligned} \frac{1}{2}A &= \frac{1}{2} \begin{pmatrix} 4 & -4 & 0 \\ 5 & 6 & 1 \\ 8 & 9 & -1 \end{pmatrix} \\ &= \begin{pmatrix} 4/2 & -4/2 & 0/2 \\ 5/2 & 6/2 & 1/2 \\ 8/2 & 9/2 & -1/2 \end{pmatrix} \\ &= \begin{pmatrix} 2 & -2 & 0 \\ \frac{5}{2} & 3 & \frac{1}{2} \\ 4 & \frac{9}{2} & -\frac{1}{2} \end{pmatrix} \end{aligned}$$

*Matrix Multiplication*Definition:

The **matrix product** of A and B , where A is of order $m \times n$ and B is of order $n \times p$ is obtained by the following prescription,

$$(ab)_{ij} \stackrel{\text{def}}{=} \sum_{k=1}^n a_{ik}b_{kj}$$

$$(1 \leq i \leq m \text{ and } 1 \leq j \leq p)$$

Note that the number of columns of A must equal the number of rows of B . To see how to use the horrible prescription, split up A into rows and B into columns; then taking the 'scalar product' of the i^{th} row of A with the j^{th} column of B gives the $(i, j)^{\text{th}}$ entry of AB .

Example 4

Calculate AB when $A = \begin{pmatrix} 1 & -2 & 0 \\ 8 & 0 & 1 \\ -2 & 4 & 6 \end{pmatrix}$ and $B = \begin{pmatrix} 3 & 0 \\ 1 & 2 \\ -3 & 4 \end{pmatrix}$.

As the number of columns of A equals the number of rows of B , the product AB makes sense.

$$\begin{aligned}
 AB &= \begin{pmatrix} 1 & -2 & 0 \\ 8 & 0 & 1 \\ -2 & 4 & 6 \end{pmatrix} \begin{pmatrix} 3 & 0 \\ 1 & 2 \\ -3 & 4 \end{pmatrix} \\
 &= \begin{pmatrix} 1 \times 3 + (-2) \times 1 + 0 \times (-3) & 1 \times 0 + (-2) \times 2 + 0 \times 4 \\ 8 \times 3 + 0 \times 1 + 1 \times (-3) & 8 \times 0 + 0 \times 2 + 1 \times 4 \\ (-2) \times 3 + 4 \times 1 + 6 \times (-3) & (-2) \times 0 + 4 \times 2 + 6 \times 4 \end{pmatrix} \\
 &= \begin{pmatrix} 1 & -4 \\ 21 & 4 \\ -20 & 32 \end{pmatrix}
 \end{aligned}$$

In general, AB is not the same matrix as BA .

Example 5

Show that $AB \neq BA$ for the matrices $A = \begin{pmatrix} 0 & -3 \\ 5 & 2 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$.

First AB ,

$$\begin{aligned}
 AB &= \begin{pmatrix} 0 & -3 \\ 5 & 2 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} \\
 AB &= \begin{pmatrix} 0 & -6 \\ 5 & 4 \end{pmatrix}
 \end{aligned}$$

Next BA ,

$$BA = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} 0 & -3 \\ 5 & 2 \end{pmatrix}$$

$$BA = \begin{pmatrix} 0 & -3 \\ 10 & 4 \end{pmatrix}$$

Definition:

When forming AB , B is **pre-multiplied** by A and A is **post-multiplied** by B .

Definition:

The **transpose** of A (with the order of A being $m \times n$), denoted by A^T (sometimes A'), is the $n \times m$ matrix obtained by interchanging the rows and columns of A ,

$$(a^T)_{ij} \stackrel{\text{def}}{=} a_{ji}$$

Example 6

Find the transpose of $P = \begin{pmatrix} 3 & -7 & 0 \\ 5 & 9 & 46 \end{pmatrix}$.

$$P^T = \begin{pmatrix} 3 & 5 \\ -7 & 9 \\ 0 & 46 \end{pmatrix}$$

Definition:

A (square) matrix A can be multiplied by itself any number of times, giving the n^{th} **power** of A ,

$$A^n \stackrel{\text{def}}{=} \underbrace{A \times A \times A \times \dots \times A}_{n \text{ times}}$$