

① Inverse Method to solve Linear Systems:

$$A \cdot X = B \rightarrow A^{-1}(A \cdot X) = A^{-1} \cdot B$$

$$\boxed{X = A^{-1} \cdot B}$$

$$[A | I] \rightarrow [I | A^{-1}]$$

Ex①: Solve the following system by Inverse Method?

$$\begin{array}{l} 5x_1 + 6x_2 = 7 \\ 3x_1 + 4x_2 = 5 \end{array} \rightarrow \begin{array}{ccc} \begin{bmatrix} 5 & 6 \\ 3 & 4 \end{bmatrix} & \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} & = \begin{bmatrix} 7 \\ 5 \end{bmatrix} \\ A & X & B \end{array}$$

$$[A | I] = \left[\begin{array}{cc|cc} 5 & 6 & 1 & 0 \\ 3 & 4 & 0 & 1 \end{array} \right] \xrightarrow{\frac{1}{5}R_1} \left[\begin{array}{cc|cc} 1 & \frac{6}{5} & \frac{1}{5} & 0 \\ 3 & 4 & 0 & 1 \end{array} \right] \xrightarrow{-3R_1 + R_2}$$

$$\left[\begin{array}{cc|cc} 1 & \frac{6}{5} & \frac{1}{5} & 0 \\ 0 & \frac{2}{5} & -\frac{3}{5} & 1 \end{array} \right] \xrightarrow{\frac{5}{2}R_2} \left[\begin{array}{cc|cc} 1 & \frac{6}{5} & \frac{1}{5} & 0 \\ 0 & 1 & -\frac{3}{2} & \frac{5}{2} \end{array} \right] \xrightarrow{-\frac{6}{5}R_2 + R_1}$$

$$\left[\begin{array}{cc|cc} 1 & 0 & 2 & -3 \\ 0 & 1 & -\frac{3}{2} & \frac{5}{2} \end{array} \right]$$

I A⁻¹

$$X = A^{-1} \cdot B = \begin{array}{cc} \begin{bmatrix} 2 & -3 \\ -\frac{3}{2} & \frac{5}{2} \end{bmatrix} & \cdot \begin{bmatrix} 7 \\ 5 \end{bmatrix} & = \begin{bmatrix} -1 \\ 2 \end{bmatrix} \\ \text{2x2} & \text{2x1} & \text{2x1} \end{array}$$

$$\therefore X_1 = -1, X_2 = 2$$

MEMO

Ex ②: Solve the system by use Inverse method?

$$\begin{aligned} x + y + 2z &= 9 \\ 2x + 4y - 3z &= 1 \\ 3x + 6y - 5z &= 0 \end{aligned} \rightarrow \begin{bmatrix} 1 & 1 & 2 \\ 2 & 4 & -3 \\ 3 & 6 & -5 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 9 \\ 1 \\ 0 \end{bmatrix}$$

$A \quad X \quad B$

$$\begin{bmatrix} 1 & 1 & 2 & | & 1 & 0 & 0 \\ 2 & 4 & -3 & | & 0 & 1 & 0 \\ 3 & 6 & -5 & | & 0 & 0 & 1 \end{bmatrix} \xrightarrow{\substack{-2R_1 + R_1 \\ -3R_1 + R_3}} \begin{bmatrix} 1 & 1 & 2 & | & 1 & 0 & 0 \\ 0 & 2 & -7 & | & -2 & 1 & 0 \\ 0 & 3 & -11 & | & -3 & 0 & 1 \end{bmatrix} \xrightarrow{\frac{1}{2}R_2}$$

$$\begin{bmatrix} 1 & 1 & 2 & | & 1 & 0 & 0 \\ 0 & 1 & -\frac{7}{2} & | & -1 & \frac{1}{2} & 0 \\ 0 & 3 & -11 & | & -3 & 0 & 1 \end{bmatrix} \xrightarrow{\substack{-R_2 + R_1 \\ -3R_2 + R_3}} \begin{bmatrix} 1 & 0 & \frac{11}{2} & | & 2 & -\frac{1}{2} & 0 \\ 0 & 1 & -\frac{7}{2} & | & -1 & \frac{1}{2} & 0 \\ 0 & 0 & -\frac{1}{2} & | & 0 & -\frac{3}{2} & 1 \end{bmatrix} \xrightarrow{-2R_3}$$

$$\begin{bmatrix} 1 & 0 & \frac{11}{2} & | & 2 & -\frac{1}{2} & 0 \\ 0 & 1 & -\frac{7}{2} & | & -1 & \frac{1}{2} & 0 \\ 0 & 0 & 1 & | & 0 & 3 & -2 \end{bmatrix} \xrightarrow{\substack{-\frac{11}{2}R_3 + R_1 \\ \frac{7}{2}R_3 + R_2}} \begin{bmatrix} 1 & 0 & 0 & | & 2 & -17 & 11 \\ 0 & 1 & 0 & | & -1 & 11 & -7 \\ 0 & 0 & 1 & | & 0 & 3 & -2 \end{bmatrix}$$

$$\therefore A^{-1} = \begin{bmatrix} 2 & -17 & 11 \\ -1 & 11 & -7 \\ 0 & 3 & -2 \end{bmatrix}, \quad A \cdot A^{-1} = I \quad ?$$

Now

$$X = A^{-1} \cdot B = \begin{bmatrix} 2 & -17 & 11 \\ -1 & 11 & -7 \\ 0 & 3 & -2 \end{bmatrix} \cdot \begin{bmatrix} 9 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 18 - 17 + 0 \\ -9 + 11 + 0 \\ 0 + 3 + 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

$$\therefore x = 1, y = 2, z = 3$$

1 Cramer's Rule

Def: Consider the system of n linear equations in n variables; $A \cdot x = B$ and $|A| \neq 0$, then

$$x_j = \frac{|A_j|}{|A|}$$

A_j = is a matrix A when replace column j by matrix B .

$|A|$ = is determinant of A .

* Determinant of matrix

1- Let $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$, Then $|A| = (a_{11} \cdot a_{22}) - (a_{21} \cdot a_{12})$

Ex: Let $A = \begin{bmatrix} 7 & 5 \\ 3 & 4 \end{bmatrix}$, then find $|A|$?

$$|A| = (7 \times 4) - (3 \times 5) = 12 - 15 = -3$$

2- Let $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$, then

$$|A| = a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

Ex ① Let $A = \begin{bmatrix} 1 & 2 & 4 \\ 8 & -6 & 2 \\ 4 & 0 & -3 \end{bmatrix}$, Find $|A|$?

So $\rightarrow$

$$|A| = 1 \cdot \begin{vmatrix} -6 & 2 \\ 0 & -3 \end{vmatrix} - 2 \cdot \begin{vmatrix} 8 & 2 \\ 4 & -3 \end{vmatrix} + 4 \cdot \begin{vmatrix} 8 & -6 \\ 4 & 0 \end{vmatrix}$$

$$= 1 \cdot (18) - 2 \cdot (-32) + 4 \cdot (24)$$

$$= 18 + 64 + 96 = 178$$

Ex ② Let $A = \begin{bmatrix} 1 & -1 & -2 \\ 1 & 3 & 1 \\ 1 & 3 & 2 \end{bmatrix}$, Find $|A|$?

$$|A| = 1 \cdot \begin{vmatrix} 3 & 1 \\ 3 & 2 \end{vmatrix} - (-1) \begin{vmatrix} 1 & 1 \\ 1 & 2 \end{vmatrix} + (-2) \begin{vmatrix} 1 & 3 \\ 1 & 3 \end{vmatrix}$$

$$= 1 \cdot (3) + 1 \cdot (1) + (-2) \cdot (0)$$

$$= 3 + 1 + 0 = 4$$

MEMO

Ex ①: Solve the system by Cramer's Rule?

$$\begin{aligned} x + 2y + 2z &= -1 \\ x + 3y + z &= 4 \\ x + 3y + 2z &= 3 \end{aligned} \rightarrow \begin{bmatrix} 1 & 2 & 2 \\ 1 & 3 & 1 \\ 1 & 3 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -1 \\ 4 \\ 3 \end{bmatrix}$$

A X B

Sol

$$|A| = 1 \cdot \begin{vmatrix} 3 & 1 \\ 3 & 2 \end{vmatrix} - 2 \cdot \begin{vmatrix} 1 & 1 \\ 1 & 2 \end{vmatrix} + 2 \cdot \begin{vmatrix} 1 & 3 \\ 1 & 3 \end{vmatrix} = 3 - 2 + 0 = 1 \neq 0$$

$$|A_1| = \begin{vmatrix} -1 & 2 & 2 \\ 4 & 3 & 1 \\ 3 & 3 & 2 \end{vmatrix} = -1 \cdot \begin{vmatrix} 3 & 1 \\ 3 & 2 \end{vmatrix} - 2 \cdot \begin{vmatrix} 4 & 1 \\ 3 & 2 \end{vmatrix} + 2 \cdot \begin{vmatrix} 4 & 3 \\ 3 & 3 \end{vmatrix} = -7$$

$$|A_2| = \begin{vmatrix} 1 & -1 & 2 \\ 1 & 4 & 1 \\ 1 & 3 & 2 \end{vmatrix} = -1 \cdot \begin{vmatrix} 4 & 1 \\ 3 & 2 \end{vmatrix} + 1 \cdot \begin{vmatrix} 1 & 1 \\ 1 & 2 \end{vmatrix} + 2 \cdot \begin{vmatrix} 1 & 4 \\ 1 & 3 \end{vmatrix} = 4$$

$$|A_3| = \begin{vmatrix} 1 & 2 & -1 \\ 1 & 3 & 4 \\ 1 & 3 & 3 \end{vmatrix} = 1 \cdot \begin{vmatrix} 3 & 4 \\ 3 & 3 \end{vmatrix} - 2 \cdot \begin{vmatrix} 1 & 4 \\ 1 & 3 \end{vmatrix} - 1 \cdot \begin{vmatrix} 1 & 3 \\ 1 & 3 \end{vmatrix} = -1$$

$$x = \frac{|A_1|}{|A|} = \frac{-7}{1} = -7$$

$$y = \frac{|A_2|}{|A|} = \frac{4}{1} = 4$$

$$z = \frac{|A_3|}{|A|} = \frac{-1}{1} = -1$$